The original publication is available at www.springerlink.com

Direct link: http://link.springer.com/article/10.1007/s11858-015-0664-9

First-Graders' Spatial-mathematical Reasoning about Plane and Solid Shapes and their Representations

David A. Hallowell[1], Yukari Okamoto[2], Laura F. Romo[3], and Jonna R. LaJoy[4]

University of California, Santa Barbara

Gevirtz Graduate School of Education

University of California

Santa Barbara, CA 93106-9490

dhallowell@education.ucsb.edu[1], yukari@education.ucsb.edu[2], lromo@education.ucsb.edu[3], jlajoy@education.ucsb.edu[4]

Abstract

The primary goal of the study was to explore first-grade children's reasoning about plane and solid shapes across various kinds of geometric representations. Children were individually interviewed while completing a shape-matching task developed for this study. This task required children to compose and decompose geometric figures in order to identify geometric shapes that either matched or did not match the stimulus shape. The stimulus shapes were 2D diagrams of plane- and solid-shape geometric figures. The results showed that children overestimated the significance of triangular vertices ("pointiness"); certain kinds of scaling demands gave children trouble in shape classification; children had trouble translating lines found in 2D diagrams into 3D visual boundaries, especially where projected curvature was involved; and that children had difficulty reasoning consistently across the task. Implications for future research as well as teaching recommendations are discussed.

Keywords Early geometry · Spatial reasoning · Dimensionality · Mathematical diagrams

MSC 97E50, 97G40, 97G80

1 Introduction

The link between early spatial reasoning ability and subsequent math achievement has been noted for decades in the literature (e.g., Guay & McDaniel 1977). However, spatial skills rarely had a place in mathematics curricula (Wai et al. 2009). At best, mathematics textbooks in early elementary grades include pictures of geometric shapes as if children could extract spatial relations within and across various shapes. What do children make of such representations? We undertook an exploratory study of first-grade children's reasoning about plane and solid shapes to answer this question.

Of particular interest was how children would reason when shown 2D diagrams corresponding to plane and solid geometric shapes. This reasoning process involves children's spatial transformation skills, which are referred to as intrinsic-dynamic spatial skills in the typology proposed by Uttal et al. (2013). Intrinsic-dynamic spatial reasoning concerns some sort of visualization of the transformation of the spatial coordinates of an object. The transformation could occur either in the object itself (e.g., imagining a circle squished into an ellipse) or in relation to a frame of reference (e.g., a polyomino rotated from an original position in space). These skills are especially relevant in sciences such as geology and chemistry, where they are regularly drawn upon in the practice of their empirical activities (Hegarty et al. 2010).

Intrinsic-dynamic spatial skills also have direct relevance to mathematics achievement. Stannard et al. (2001), for example, found a relation between the complexity of construction play with blocks (LEGOs) and a carpentry activity measured in preschool, and performance on standardized mathematics achievement tests in grade 7. In a longitudinal study, Gunderson et al. (2012) found that spatial transformation ability at age 5 predicted performance on approximate numerical, not quantity (dot), comparisons measured at age 8. They further found that this relation was mediated by the linear number line knowledge measured at age 6. The researchers suggested that the number line served as a spatialized representation of symbolic numeracy, which supported the arithmetic advantage shown by those individuals whose spatial reasoning skills were advanced. These studies represent only a small selection of work showing a positive relation between spatial transformation skills and mathematical achievement.

2 Theory and Prior Research

2.1 Children's Interpretations of Geometric Representations

Children are exposed to geometric shapes from early years. For example, teachers hang posters on the classroom walls depicting plane and solid shape. Textbooks and worksheets alike show representations of 3D shapes on a 2D surface. Educators, perhaps not unreasonably, expect that such depictions are communicating the spatial-mathematical mental-referents that they wish to discuss with their students. However, research on children's reasoning about mathematical diagrams suggests that children's interpretations of such representations are not made in a straightforward manner (e.g., Steenpaß and Steinbring 2013). To quote Lonergan (1997), human knowing is not "some sort of metaphysical sausage machine" (p.34), where images travel into the brain and pop out as conceptual knowledge. Mathematical insight marshals complex cognitive activities to obtain understanding, and the potential pitfalls are myriad.

Another potential sense-making tool for children in the world of mathematical representations are diagrams of shapes. Building on the work of van Hiele (1986), Clements et al. (1999) conducted clinical interviews with 97 middle-class children 3.5-6.9 years old on a shape-sorting task. The goal was to investigate how children classify and discriminate shapes. For each item, children were asked to identify a circle, square, rectangle, or triangle among simple line drawings of shapes on a white sheet of paper. The activity required children to discriminate target-class shapes from non-class shapes, as well as to identify nonstandard examples of shapes (e.g., a tall, skinny triangle with a severely acute angle). Standard shapes also varied in sizes (e.g., a circle with a diameter of 0.5 inch and another with a diameter of 2 inches). These researchers sought to ascertain the extent to which children's shape identification relied upon visual- versus property-based modes of analysis. Examples of visual explanations included responses oriented at the object's perceptual appearance, such as "skinny/fat/long," "pointy corners," or "'sort of' like." Property-based explanations related to aspects such as "round/no sides," "number of corners," or "number of sides."

For all three age groups (ages 4, 5, and 6), Clements et al. (1999) found that between 84% and 98% of responses were either no-response, “I don’t know,” or “just because.” Six year-olds outperformed 4 and 5 year-old children. Mean accuracy scores were highest for circles (14.4/15 possible) and squares (11.3/13 possible), followed by triangles (8.2/14 possible) and rectangles (8.2/15 possible). A visually dense array of circles and squares that required considerable disambiguation skills yielded the lowest accuracy rate (17.2/28). Children were less accurate with nonstandard than standard target shapes. The percentage of children giving no response, “I don’t know” or “just because” at least once for each item was over 50% for every shape and age group. This was the case for circles and squares where children’s correct identification rates were higher than the other shapes. They also found that children were more likely to rely on visual explanations than property explanations, but that accuracy was positively correlated with the number of property explanations. Children who made use of the relational, property-depicting aspects of these geometric diagrams were able to use them more effectively than those who simply focused on the visual, object-based aspects of these shape depictions. Additionally, children seemed to have difficulty expressing the reasons behind their choices, even when justifying accurate class-selections they have made. Although some children might have had a reliable intuitive ability to accurately classify shapes, still fewer children were sufficiently aware of their own processes for shape classification to be able to explain their own reasons for such judgments.

2.2 Cognitive Acts and Geometric Representations

For researchers and educators to be able to effectively grasp the challenges that children face when learning from these pedagogical tools, it is important to have an explanatory theory of the cognitional acts that underlie these learning triumphs and tragedies. What is also needed, though, is a theoretical basis from which to begin to account for the cognitional activities of a child who is making sense of this or that mathematical diagram. Understanding these cognitive activities should allow us to identify why children struggle to have the intended insights in certain kinds of representations, yielding clues as to how pedagogical experiences may be structured more effectively to facilitate student learning.

Duval (2006) provides a developed framework for understanding cognitional activities in mathematical learning contexts. He pointed out that educators and researchers often sidestep the issue of the cognitive activities required for learning in mathematics. Instead, they focus on either conceptual understanding or acts of mathematical understanding that require minimal semiotic transformation across representations. Approaches lacking an underlying cognitional theory are insufficient, because they ignore the underlying semiotic operations that are required for deep mathematical learning to take place. He argued that mathematics is an area that presents special difficulties in education. This is because the mathematical objects are never directly shown in any given representation, but their apprehension also cannot be obtained without those very representations. Students never “see” the complete mathematical object “square rectangle” in any single diagram. Instead, they come to understand the common properties related to square rectangle after encountering many visual examples of the object, as well as contrasting those properties with other kinds of rectangles, etc. This situation creates unique pedagogical challenges.

Earlier, Duval (1995) discussed his analysis of students’ thinking while sense-making with geometric representations and addressed why there is an “underlying cognitive complexity of even the simplest geometrical figures” (p. 143). His analysis looked at cognitive apprehensions to explain why geometric diagrams function as mathematical heuristic representations for some individuals and not others. He outlined four types of apprehensions: 1) *Perceptual apprehensions*, which concern the perceptual constancies and Gestalt patterning of experience when we visually scan a representation; 2) *Sequential apprehension*, when we use tools and technical precepts to construct a represented mathematical object; 3) *Discursive apprehension*, where the characteristics and meaning of a geometric representation are constrained through statements concerning the representation; and 4) *Operative apprehension*, which involves visualizing various spatial transformations to gain insight into the solution of a given problem. A representation must involve a perceptual apprehension plus 1 of the other 3 apprehension categories to function as a mathematical heuristic representation.

Duval (1995) suggested that, in the mode of operative apprehension, there are three ways that cognitional acts may function to modify a given representation to yield insight: In the *mereologic way*, the individual visualizes the figure in component shapes to either reorient external borders of the representation, or to create new sub-shapes. In the *optic way*, one morphs the shape by making it wider or narrower, shorter or taller, etc. Finally, the *place way* concerns changing the representation’s overall orientation on some larger frame of reference. Duval’s (1995) research shows how the particular structure of a geometric representation may drastically inhibit a student’s effective use of a given figure as a heuristic tool. As geometric representations require a greater quantity of operative apprehensions to unveil the critical mathematical insight, the number of students who are able to use them for learning plummets. He conducted a study where 120 students ages 13-14 were asked to complete problems based on three different geometric diagrams, comparing a shaded to an unshaded portion of the geometric figure. The diagrams all related to the same underlying geometric concept, but were structured to require increasing numbers of operative apprehensions. In each case, students were shown a diagram of a parallelogram divided in various ways by shaded and unshaded regions, as well as by lines. For example, in a simpler version, a single unshaded triangle extends with its base along one side of the parallelogram. The remaining space is shaded. In a complex diagram, a trapezium has two unshaded triangles extending from the base and top of the diagram to a common point, and two shaded triangles extend from the sides. A student must use the same lower triangle twice as part of two distinct triangles in the larger image to make the necessary geometric inference.

Duval (1995) found support for the hypothesis that students' success rates varied significantly depending on the number and complexity of operative apprehensions required by the diagram. For students to visualize the necessary symmetry implied in the unshaded regions, several part-whole reconfigurations were necessary. Depending on the diagram presented to students, accurate solution rates varied from 60% to 0%. Diagram characteristics that inhibited performance in his study were operations such as having to visualize splitting in embedded sub-figures, comparison regions that lacked complimentary configurations, figures that are not convex, and especially figures requiring double use of a visualized region. When geometric diagrams required complex operative apprehension to obtain a critical visualization for resolving a solution, the percentage of students who had access to the representation as a tool fell. Duval expressed concern in the need for children to develop these important visualization skills, proposing that this might be done also by using demanding figures during teaching.

2.3 Young Children's Understanding of Representations

A recently adopted national standard in the United States recommends first-graders conduct shape composition across 2 and 3 dimensions (CCSS.Math.Content.1.G.A.2). However, it is difficult to find clear guidelines for teachers regarding how such an early geometry experience might be implemented for optimum student learning. Prior studies about young children's understanding of shape have relied on paper-and-pencil measures or on non-stereoscopic images depicted on 2D computer monitors (e.g., Clements et al. 1999). Herbst (2004) analyzed how older children use geometric diagrams in reasoning conjectures, emphasizing the referenced geometric object in discursive reasoning in geometry. However, his analysis was primarily concerned with 2D diagrams. Thus, it remains unclear from studies employing these kinds of 2D measures how children make sense of geometric representations when 3 dimensions are involved in learning activities. In this study, we put 3D manipulatives of plane and solid shapes in the hands of children and asked them to reason about these shapes when shown 2D diagrams of related shapes. Our aim with this exploratory study was to begin to discriminate trends in how children make sense of plane and solid shapes and their representations so that future research and educational interventions have some early empirical clues regarding promising avenues of work. From a mathematical cognitional perspective, this study was concerned primarily with how young children spontaneously made use of perceptual and operative apprehension modalities after receiving minimal verbal instructions to structure a shape composition/decomposition task.

3 Method

3.1 Subjects

Participants were recruited from a large charter school in the California Central Coast region with characteristics similar to statewide demographics (47% White, 46% Hispanic, 3% Asian, 4% other). Over a third of students at the school were eligible for free or reduced-price meal assistance (40.13%). All students who returned signed consent forms were invited to participate. Participants for this exploratory analysis were 36 first-graders (N = 36, 20 females, M_{age} = 6.85, 16 males, M_{age} = *6.96*, total M_{age} = 6.90, age range: 6.14-7.40 years).

3.2 Materials and Procedures

Participants were individually interviewed. Each interview followed a semi-structured interview protocol, was video-recorded, and lasted approximately 30 minutes long. Children completed a shape-matching task that was designed for this study to elucidate children's spatial-mathematical reasoning about shapes across various 2D and

3D representations. Interviews were conducted on-site in a quiet corner where students were accustomed to working. Participants were reminded that they could terminate a task or the interview process itself at any point during the interview. Two graduate students carried out all of the interview sessions. One served as a primary interviewer while the other assisted the first and took detailed notes. Images from a popular children's geometry cartoon were displayed at the far edge of the table, and the interviewer established rapport with each child through a brief story about exploring shapes in Shapeland. Children were invited to accompany the interviewer on an exploration of Shapeland, and offered a pith helmet to wear during the exercise if the child desired it. The interviewer also wore a pith helmet whenever the child chose to do so.

3.3 Plane- and Solid-Shape Matching Task

A 10-item shape composition-decomposition sorting task was developed for this study (two items were dropped from the analysis for reasons described below). Additionally, a training item was administered first to familiarize the participants with the task. We built on the prior work of Clements et al. (1999) by incorporating shape composition and decomposition activities into the task itself. We sought to extend their work by moving target objects from 2D representations to 3D representations. Stimulus items were presented in 2D because these are typically the kind of representations that children encounter on posters, textbooks, and worksheets. We reasoned that having 3D manipulatives as target items might help students explain more of what they understood about solid shapes than if they had 2D depictions of solid shapes with which to match and reason. Duval's (1995) modes of operative apprehension were theorized to underlie the shape composition and decomposition activity, since visualizing how shapes break down into component shapes or combine to form composite shapes required envisioning part-whole relations, scaling activities, and alternate perspectives.

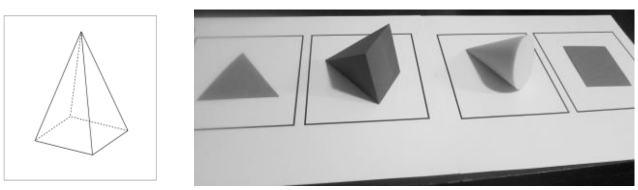

Figure 1. The 2D solid-pyramid stimulus-item and 4 3D shape manipulatives. The direct match is represented by the solid pyramid, whose actual dimensions were very close to those of the diagram. The flat triangle and flat square are challenge items, and the solid cone is the distractor.

For each item, the child was shown a 2D stimulus item printed in black ink on a 5x5-inch card on a white background. Five 2D plane-shape items (triangle, rectangle, circle, square, and hexagon) and five 2D solid-shape items (cube, sphere, rectangular prism, pyramid, and cylinder) were presented to children (see Figure 1 for an example item). Plane-shape items featured simple, orthogonal diagrams depicting shapes whose points could all be located on an x-y coordinate system. Solid-shape items showed isometric depictions of shapes whose referents require an x-, y-, and z-axis coordinate system to map every point upon. Four physical (3D) shape manipulatives were placed on a white background in front of the child. Plane-shape manipulatives were flat, paper-thin manipulatives. Solid-shape manipulatives were foam, wood, and plastic solids small enough to fit in a child's hand. The order and orientation of placement was standardized for each administration of the task. Care was taken so that solid-shape manipulatives were oriented toward the child to reveal multiple faces (e.g., pyramid manipulative resting at an angle so that square base and triangular face were both exposed to the child, see Figure 1.)

Objects	Exact Match	Target Objects (Challenge Matches)		Distractor Object
2D Plane Stimulus	• Match Requirements	• Match Requirements	• Match Requirements	• Exclusion Requirements
Triangle	Flat Triangle: • Direct translation	Pyramid: • Ignore square base • Scale and match to triangular face	Cone: • Ignore circular base • Scale and match to triangular face	Octagonal Prism: • Disambiguate octagonal and rectangular faces from stimulus

Square	Flat Square: • Direct translation	Rectangular Prism: • Ignore non-square rectangular faces • Scale and match to square faces	Pyramid: • Ignore triangular faces • Scale and match to square base	Hexagonal Prism: • Disambiguate hexagonal and non-square rectangular faces from stimulus
Circle	Flat Circle: • Direct translation	Cylinder: • Ignore rectangular face • Scale and match to circular faces	Cone: • Ignore triangular face (especially "pointiness" of apex) • Scale and match to circular base	Hexagonal Prism: • Disambiguate hexagonal and non-square rectangular faces from stimulus
Rectangle	Flat Rectangle: • Direct translation	Rectangular Prism: • Ignore square faces • Scale and match to non-square rectangular faces	Triangular Prism: • Ignore triangular faces (especially "pointiness" of apex) • Scale and match to rectangular face	Cone: • Disambiguate circular and triangular faces from stimulus
2D Solid Stimulus				
Pyramid	Pyramid: • Direct translation	Flat Triangle: • Notice shape outline • Scale and match to triangular pyramid face	Flat Square: • Notice shape outline • Scale and match to pyramid base	Cone: • Disambiguate single triangular face from delineated triangular faces on stimulus • Distinguish circular base from square base
Cube	Cube: • Direct translation	Large Cube: • Scale shape to stimulus	Flat Square: • Notice shape outline • Scale and match to cube side	Small Rectangular Prism: • Disambiguate square-rectangle from non-square rectangle
Rectangular Prism	Rectangular Prism: • Direct translation	Small Rectangular Prism: • Scale sides to match stimulus side-length ratio • Scale and match shape to stimulus	Flat Rectangle: • Notice shape outline • Match to rectangle face	Triangular Prism: • Ignore rectangular face on distractor • Disambiguate triangular face on distractor from rectangular face on stimulus
Cylinder	Cylinder • Direct Translation	Flat Rectangle: • Notice rectangular face • Scale and match to rectangular cylinder face	Flat Circle: • Notice shape outline • Scale and match to circular face	Cone: • Ignore circular faces on stimulus and distractor • Disambiguate rectangular face of stimulus from triangular face of distractor

Figure 2. Overview of Plane- and Solid-Shape Sorting Task. Scaling requires Duval's (1995) optic way of opertional apprehension. Matching and disambiguation are reliant upon the mereologic way of operative apprehension. Direct translation and noticing shape outlines rely upon perceptual apprehensions.

For each item there was one distractor item and three target matches. Wherever possible, the item was structured so that there was one direct 3D match to the 2D stimulus and two distinct challenge target items that required shape composition or decomposition depending on the nature of the stimulus (see Figure 2 for an overview of the task). The number of target objects was chosen with this item structure in mind. Plane-stimulus challenge items often required children to compose a shape to make a match, and solid-stimulus challenge items often required them to decompose shapes to correctly identify target items. In some cases, such as with the solid-cube stimulus item, there was insufficient variation in the shape itself to emulate this pattern, so challenge items required children to scale faces to correctly identify a match.

The training task was designed and administered for the plane-shape stimulus type only. This decision was made because we were particularly interested in children's spontaneous reasoning about solid shapes for this exploratory analysis. For the training task, a plane octagon was shown to children. A flat octagon and a solid octagonal prism accompanied a flat circle and a solid cylinder. Children were read the following instructions: "Here is an image of a plane shape. Plane shapes sometimes get put together to make solid shapes [Hold up example of solid shape]. There can be more than one shape that has this shape in it. Also, sometimes this shape may be tall and skinny or short and fat. Can you find all of the shapes in front of you that match the image?" After children correctly identified matches on the target item, plane-shape stimulus items were administered first.

Next, the interviewer administered the solid-shape stimulus items. The following instruction was provided: "Here is an image of a solid shape. Sometimes you can find a solid that matches the shape on the card, and sometimes you can find shapes that match parts of the solid shape. Also, sometimes this shape may be tall and skinny or short and fat. Can you find all of the shapes in front of you that match the image, or are parts of the shape in the image?"

For each item of the shape-sorting task, the interviewer inquired about the reason for at least one match and one exclusion. Whenever matching or exclusion errors were made, the interviewer inquired about each error for that item. This was done to provide a range of responses that children gave regarding their spatial-mathematical reasoning, especially in those instances where children had difficulty correctly matching or excluding shape manipulatives.

4 Results

4.1 Coding and Overview

Videos of children's performance on the shape-sorting task were coded into The Observer 11 (Noldus Information Technology 2008) and subsequently exported into SPSS 22 (IBM Corporation, 2013) for descriptive quantitative analysis. To determine if girls and boys differed in their overall performance, we carried out a Mann-Whitney U-test. The result showed no gender differences. Thus the data from girls and boys were combined for the subsequent analyses. We structured our analysis to identify the most common errors associated with the task, and to identify common explanations children gave for erroneous exclusions of matches and inclusions of distractors. We also discussed noteworthy observations from our qualitative analysis.

After conducting a preliminary analysis, 1 plane-shape stimulus item and 1 solid-shape stimulus item were dropped from further analysis. The plane-stimulus hexagon item included an octagonal prism as a distractor item, which was visually similar to the target-match hexagonal prism. Children counted the sides to exclude the distractor item, an exclusion skill not consistent with other items in the measure. Because we are primarily interested in children's reasoning about dimensional, compositional and decompositional reasoning about shapes, the decision was made to drop the item from analysis for measure coherence.

Additionally, the solid-shape sphere stimulus item was dropped from analysis. A distracter item was a solid-hemisphere manipulative. This manipulative had yielded interesting interview data in an early pilot test with a younger child, but after analyzing children's reasons for including it as a match in this study, it was decided that the wording "sometimes you can find shapes that match parts of the solid shape" in the solid-shape instructions led children to cogently identify the shape as a match. The remaining 8-item measure is the one analyzed in this work.

Children's responses were coded in two different ways. For the first coding scheme, we created a tally of accurate responses for each target object across the 8 items. The 11 target objects with the highest error rates (over 20% of children erred) are presented in Figure 3. For the second coding scheme, we coded justifications for errors. Explanations were initially coded by semantic content in order to provide a descriptive analysis of the patterns that emerged from children's responses. Categories were developed based on content directly featured in children's statements to avoid ambiguity in interpretation. We further identified children's explanations corresponding to Duval's (1995) components of operative apprehension. Explanations were categorized as related to mereologic, optic, or place reasoning where relevant. Table 1 provides an overview of the categories and coding scheme Since these are associated with errors, they represent an oversight of the corresponding strategy. Figure 3 shows the frequencies of children's explanations for the most common errors.

Table 1. Overview of how children's justifications for errors on the matching task were classified.

Overview of Categorization of Children's Match and Exclusion Error Justifications		
Category	**Criteria**	**Example**
Sides	Child mentioned sides or edges	"because it has (4) sides"
Angles	Child cited "pointiness" of object in his or her explanation, often pointing at external angle of point	"because it's pointy," "it has a point," etc.
Mereologic	Child pointed out 1 part of a shape while ignoring another key shape part critical for accurate classification or Justified not making an appropriate part-whole distinction	"they both have circles" (referring to the circular face of a cylinder and cone) "this part has a point and circles never have points" (evaluating a plane circle and a cone)
Optic	Child referred to scaling aspects of objects for differentiation or Cited a characteristic referring to volume	"it's too small," "it doesn't fit" "because it's round" (referring to projected *z*-axis of surface of object)
Place	Child appealed to a transformed frame of reference	holds target object up to stimulus cites rotated perspective as a justification
Analogy	Child cited a non-geometric figure with shared geometric qualities	"it's like a tent"
NR	No gave no response	
SNE	Child cited a shape name without further explanation	"it's a triangle"
IDK	Child states that he or she does not know	"I don't know."

Children in this study identified direct shape matches for both plane and solid shapes with perfect accuracy. Frequently, children held the target object directly up to the stimulus diagram and stated something to the effect of, "see, it's the same." These first-grade children seemed to understand the representational nature of the shape diagrams depicted on the stimulus cards without difficulty. Their accuracy on the distractor objects was without error where the distractor object did not share at least 1 matching face with the stimulus item. This accounted for their perfect accuracy on plane-shape distractor items, since it was not possible to feature a target item intended for exclusion that shared a face with the stimulus. Such an object would then constitute a match according to the instructions we gave them. For the solid stimulus items, errors were made by at least some children on all distractor objects since each one shared at least 1 common shape face. A paired-samples t-test comparing performance on plane-shape items (*M* = 13.11, *SD* = 1.28) vs. the solid-shape items (*M*= 10.89, *SD* = 1.41) showed that children performed significantly better on the plane-shape items than the solid-shape items, $t(35) = 8.40$, $p < .001$.

4.2 Troublesome Triangles

Children had the most difficulty when finding the rectangular face of a triangular prism for a plane-rectangle stimulus. The triangular prism featured 3 rectangular faces situated between 2 triangular faces (Figure 4). None of the 36 children in this study was able to successfully match the plane rectangle to the rectangular face of the triangular prism. In justifying their exclusion errors, children gave the highest number (15) of no response/shape with no explanation/I don't know responses (NR/SNE/IDK) for any error. Children cited the number of sides as the next most frequently-given justification (8), which children often stated was "3 sides." One child counted 6 vertices on the triangular prism, and explained that the plane rectangle stimulus only had 4. Despite the fact that the 3 rectangular faces of the triangular prism each boasted a surface area 2.5 times larger than each of the 2 triangular faces, the triangular faces somehow absorbed all of the children's attention. Though optically the rectangular faces appeared long, the actual dimensions were close to a square rectangle, and it is possible that some students excluded the target because the stimulus was a non-square rectangle. No students gave this justification when asked why they excluded the item.

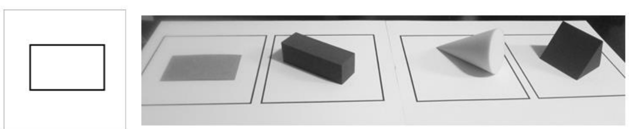

Figure 4. The 2D plane-rectangle stimulus-item and 4 3D shape manipulatives. The direct match is represented by the flat rectangle. The rectangular-prism and triangular-prism solids are challenge items, and the solid cone is the distractor. No child was able to match the rectangular face of the triangular prism (far right) to the stimulus, which required ignoring the triangular faces.

Another visualization that children had particular difficulty with was the mereological operation of noticing the relation between the plane square and the solid pyramid, which featured a square base. When the stimulus was a 2D depiction of a plane square, 12/36 children failed to include the target match. This error rate increased dramatically when the stimulus was a 2D depiction of a solid pyramid and a flat square manipulative was the target match (Figure 1), with 28/36 children erroneously excluding the item. Again, in both items the saliency of the triangular face of the pyramid seemed to cause children difficulty: Ten children either cited a shape name without further explanation (SNE, e.g., "it's a triangle") or angle (e.g., "it's pointy") for the plane-square item, and 16 contrasted the square with the pyramid in the same way for the pyramid stimulus item. The higher error rate for the solid-pyramid stimulus item may have been due in part to the orientation of the pyramid featured in the 2D diagram, which oriented the square base at the bottom of the card. However, for the plane-square stimulus item the pyramid target was presented in such a way as to reveal both the square and triangular faces to the child. Many children immediately grasped the apex of the pyramid, setting it down on the square base and proceeding to err on the item. The pointed feature of the pyramid was enough on its own for some children to reject the target as a match. Some children did not explore the targets thoroughly enough to see all the parts of the whole. Others did not know where the relevant parts were on the surface of the whole to make an accurate match. Across the 2 items, 7 children rejected the match because of the quantity of sides. Two children referenced shape analogies in their justifications for the plane-square item, one stating that the pyramid "looks like a teepee," and the other pointing out that "it looks like the top of a house."

Triangular features also gave children difficulty on the plane-circle item when deciding whether or not a cone manipulative featured a circle on one of its faces. Fifteen of 36 students rejected the cone, with 6 justifications given in the angle category, and 5 in the mereologic category. For the latter group of justifications, children pointed out that the cone had a circular face, but the apex of the cone was unacceptable for the presence of a match. One child stated, "It could be a match because it has a circle on the bottom, but it has a point… and circles never ever have points…. even an oval, they don't have points, they're just curved." Although they could remember to decompose shapes on other items, the pointy triangle on the cone might have been too egregious a difference to allow for a part-whole comparison. The points of triangles seem to function as irresistible banners for children of this age, calling for overinvestment in their significance on this kind of task.

Undertaking the plane-triangle stimulus item, 23 children incorrectly excluded the solid-cone manipulative. A plane triangle was projected onto the face of a solid cone, but this was difficult to envision unless a perspective was taken where the solid cone was rotated on its horizontal axis until its base visually depicted a straight line from the perspective of the viewer. Children struggled with the dimensionality of the problematic target item, with 4 children noting that the cone did not have "sides like that one," referring to the 3 straight lines comprising the 2D diagram of the plane triangle. One possibility is that these children had difficulty mentally rotating the face of the cone so that its "roundness" diminished in emphasis, revealing the 3 straight outside edges they associated with the "straight lines" in the plane triangle diagram. Oddly (and perhaps revealingly), in this one case where it would have been helpful to focus on the pointiness of this stimulus in making a decision to match, children ignored the point and cited the circular face in 8 of the 23 errors. It is possible that children were attending to the most disparate features on these types of tasks when deciding on a match, with "pointy" and "round" functioning as the highest-contrast features available.

4.3 Inclusion vs. Exclusion Errors

The majority of commonly made errors by children for this task were exclusion errors (8/11 errors). That is, they incorrectly excluded a shape that was intended as a match. Three of the 11 errors were inclusion errors, where the target was mistakenly included. We discussed earlier the constraints that spatial reality placed on the kinds of distractor items we could use for plane- vs. solid-stimulus items.

For the solid rectangular-prism stimulus, 22/36 children incorrectly excluded the solid short rectangular prism manipulative. This target object presented a challenge in that the shape was compact enough to be mistaken for a cube, the ratio of the non-square rectangular sides was different from that of the stimulus item, and the longest edge of the target was less than half the length of the figure in the 2D stimulus. The prevailing category related to children's oversight on this item was the optic category (11). Because the ratio of the side lengths of the target item differed from that of the stimulus item, the short rectangular prism did not immediately appear to be a shrunken version of the stimulus. A number of children held the target up to the stimulus, and observing its half-length along their common *x*-axis, rejected it as being too small to co-classify. Another 9 children simply gave an NR/SNE/IDK.

Children also had difficulty scaling where the solid cylinder stimulus was concerned. Only 2/36 children correctly identified the complete match-exclusion pattern. Sixteen children incorrectly excluded the flat circle manipulative from the solid-cylinder stimulus item, with 7 children citing justifications representing an oversight of optic operations. The reasons given related either to the circles being too far apart in diameter to share a class or to the "round" character of the cylinder (i.e., projected points on the *z*-axis). Visualizing the circles at some shared mean diameter or imagining the cylinder collapsed onto 2D taxed their optic apprehension abilities extensively.

Stimulus	Errors: Chosen stimulus (Error type/error %)	Match or Exclusion Requirements	Reasons Children Gave for ID by Category [More than 1 per ID possible]						
			Sides	**Angles**	**Mereologic**	**Optic**	**Place**	**Analogy**	**NR/SNE/ IDK**
Plane Rectangle	Solid Triangular Prism (Exclusion/100%)	• Ignore triangular faces ("pointiness" of apex) • Scale and match to rectangular faces	8	4	1	5	1	3	15
Solid Pyramid	Plane Square (Exclusion/77.8%)	• Notice shape outline • Scale and match to pyramid base	4	7	3	4	1	0	9
	Solid Cone (Inclusion/41.7%)	• Disambiguate single triangular face from delineated triangular faces on stimulus • Distinguish circular base from square base	0	10	3	5	0	1	5
Solid Cylinder	Plane Rectangle (Exclusion/72.2%)	• Notice rectangular face • Scale and match to rectangular cylinder face	6	5	2	6	2	1	12
	Plane Circle (Exclusion/44.4%)	• Notice shape outline • Scale and match to circular face	0	0	2	7	1	1	6
	Solid Cone (Inclusion/27.8%)	• Notice cone apex and lack of triangular face on cylinder	0	4	13	1	1	0	1
Solid Cube	Solid Short Rectangular Prism (Inclusion/69.4%)	• Disambiguate square-rectangle from non-square rectangle	2	0	4	12	1	0	7
Plane Triangle	Solid Cone (Exclusion/63.9%)	• Ignore circular base • Scale and match to triangular face	4	8	11	9	0	0	3
Solid Rectangular Prism	Solid Short Rectangular Prism (Exclusion/61.1%)	• Scale sides to match stimulus side-length ratio • Scale and match shape to stimulus	1	1	0	11	0	1	9
Plane Circle	Solid Cone (Exclusion/41.7%)	• Ignore triangular face ("pointiness" of apex) • Scale and match to circular base	1	6	5	6	1	0	2
Plane Square	Solid Pyramid (Exclusion/33.3%)	• Ignore triangular faces • Scale and match to square base	3	6	1	2	1	2	4
		Totals by Category	29	51	45	63	9	9	73

Figure 3. Descriptive overview of children's accuracy and reasoning across geometric representations on the shape-matching measure. As in Fig. 2, scaling is associated with optic apprehensions; matching and disambiguation require mereologic apprehensions. Direct translation and noticing shape outlines are perceptual apprehensions.

This phenomenon also came into play where children reasoned about the solid-cube stimulus, where 25/36 children incorrectly included the solid short rectangular-prism manipulative as a match. The distractor in this case featured short, non-square rectangular faces, and was about half the volume of the solid-cube stimulus item, possibly leading children to employ a sort of visuospatial shorthand, quickly scaling the ratios of the distractor sides to fit the square-rectangular faces they were referring to on the stimulus diagram. Fourteen children gave reasons whose semantic content indicated they quickly performed optic operations to visualize a match to the stimulus, representing an oversight of mereologic operations. Children who correctly excluded this target had to stop and take the time to differentiate the part-whole relations to distinguish the non-square rectangular faces on the short rectangular prism from the square rectangles on the cube. One child who made this error even proceeded after noticing the difference, stating, "because the top [matching faces of stimulus and target] and it's a little bit smaller [non-matching faces] but it's still close enough."

4.5 Dimensionality and Spatial Reasoning

Children also had difficulty with the solid-cylinder stimulus item and the flat rectangle. Twenty-six children failed to match accurately. There were several possible ways of optically apprehending the relations between the features of the stimulus and the target for these representations, and this is reflected in the heterogeneity of the justifications given. The cylinder is an object whose curvature was overwhelmingly salient for the children, and the rectangle was contrasted by its straight lines and corresponding angles. Children often referred to the "pointy" corners of the rectangle when distinguishing it from the stimulus. Some children demonstrated advanced, topological reasoning in their justifications, bending the rectangle manipulative into a roll and explaining that it could be a match if you were allowed to perform the relevant physical translations. They seemed to grasp the optic apprehension that the rectangle could be curved to "map onto" the voluminous surface of the cylinder.

Indeed, there are multiple ways one might visualize the rectangular face of a cylinder with the z-axis collapsed down onto an x-y coordinate system. One might appeal to a mental simulacrum of the cylinder, mentally shrinking its projected nature into the target object. This manner of optic apprehension was somewhat sophisticated, insofar as points on the projected surface collapsed at varying distances depending on their location on the curve of the projected surface. It was not clear that children had this ability, and seemed unlikely that they did. Another possible way to envision this relation was to view the manipulative in such a way that its secrets were revealed. For the plane-triangle stimulus item, children had difficulty envisioning the face of the cylinder rotated to a view level with the eye-line, where the outside edges of the shape formed the "sides" of the flat rectangle. Such a reorientation of perspective would circumvent the need for optic apprehension, instead relying on perceptual apprehension. Children in this study seemed to be unaware of this strategy. When explaining their exclusion errors, 6 children cited the 4 sides of the target in comparison with the rectangular face of the cylinder explaining that the cylinder "doesn't have sides," and another 5 cited the angles formed at the corners of the rectangle as distinguishing it from the cylinder. The projected volume of the manipulative seemed to prevent children from translating the visual boundaries of the object to the lines represented in stimulus diagrams.

4.6 Reasons by Category

As with the prior work of Clements et al. (1999), NR/SNE/IDK constituted the most frequent category of response given by these first-grade children, with 73 such responses. Our study indicates that even when children were given the opportunity to explain their reasoning about 2D geometric diagrams with 3D manipulatives in hand, they still had difficulty articulating systematic reasoning about the relational elements in the diagram that define sets of shapes across irrelevant transformations and combinations. These first-grade children seemed to be using salient key features of shape classes to perform on the matching task, and their metacognitive awareness of these visual processes was not readily expressed, if it existed at all. Most often on this shape-matching task, errors occurred because children simply did not know what criteria were important for the judgment at hand.

The second greatest area of oversight was related to errors in optic operations, with 63 instances related justifications given. Even though the task directions advised them otherwise, children showed difficulty with the notion that small and large or short and long examples of a given shape still belonged to the same class. Children often rejected like-shapes due to these accidental disparities. Also, when evaluating the relation between a plane shape and a solid shape, children often had difficulty imagining the projected dimension of the z-axis collapsed down onto x and y coordinates. Making the link between the representational lines of a geometric diagram and the external visual boundaries of a projected surface was also very difficult for them.

The category with the third greatest number of responses (51) was the angles category. Occasionally these responses referenced vertices of rectangles, but the vast majority of these explanations referenced either a "pointy" object or directly the apex of a triangle. In the case of the triangular prism, where none of the children correctly matched the plane rectangle stimulus to the rectangular faces of the prism, the overall appearance of the prism gave an overwhelming impression of "triangle." While mereologic explanations were the fourth-most common category (45), it is likely that many of these angle responses also related to an inability to decompose the whole into parts. To rate responses consistently, we were compelled to report them separately, but in reality these areas of reasoning were often interrelated. To visualize the much larger rectangular faces of the triangular prism, children had to abandon their initial impression of the whole object and visually map the component faces as distinct shapes. They were simply unable to do this when the original impression of the whole was a feature as salient as a triangle.

5 Discussion

Our shape-matching task required children to successfully identify correct matches across representational modalities, interpreting 2D depictions of geometric shapes and mapping those depictions onto physical objects, and vice versa. Once children decided on the key features to attend to for each stimulus diagram, they then had to decide which features to focus on and which to ignore in deciding which target items fit the task requirements. Although a small sample of children from a single elementary school in California cannot be claimed to be representative, these descriptive data do show that some first-grade children can effectively reason across 2D diagrams of plane- and solid-shape geometric figures with 3D manipulatives representing those shapes. Clear patterns in the data emerged that suggest preliminary implications for teachers and designers of educational materials. These data also suggest possible lines of inquiry for future studies about children's reasoning across various kinds of geometric representations.

Our findings support earlier work showing children had difficulty explaining their reasons for why they made a particular classification decision (Clements et al.1999). Where they were able to articulate justifications for errors, oversights of mereologic and optic operations constituted a substantial portion of the errant reasoning reported. The presence of angles were a highly-salient feature that, though treated separately here, often created difficulties for children when considering part-whole shape comparisons. Children also exhibited difficulty linking the external visual boundaries of projected surfaces to the edge lines structuring shape diagrams.

5.1 Future Research

5.1.1 Sets of Feature Comparisons

It is possible that children were particularly drawn to counterexamples, such as in the case of the plane-triangle stimulus discussed in section 4.2. Across items, children's attention was often inordinately drawn to the "pointiness" of triangles. However, when children were required to project a 2D image of a plane triangle onto the face of a solid cone manipulative to accurately match the item, they actually ignored the matching points on the stimulus and the target. Instead, they attended to the circular base of the cone manipulative when incorrectly excluding the shape. It may be that counterexample characteristics of 3D shapes are more visually salient to children than matching characteristics in their spatial-mathematical reasoning. This may be particularly true when children are drawing upon their visuospatial system to decide which features they privilege during acts of judgment in the matching task. Also, particular sets of feature comparisons may override other relevant comparisons when decision-making, such as the comparison between vertices and curved surfaces. Future research designed to highlight the moment of decision for like and unlike characteristics would be helpful in understanding how children prioritize feature comparisons when making spatial-mathematical judgments.

5.1.2 The Structure of Diagrams

As discussed above, Duval (1995) has shown that how a geometric graphic is constructed can bear considerable influence on children's ability to effectively use that diagram as a sense-making tool. Thom and McGarvey (this issue) draw upon the embodied cognition perspective in pointing out that who creates the diagram matters. When students construct geometric diagrams for themselves, an increase in geometric awareness naturally ensues. Much of the time, as in the in the case of the stimulus items in this study, these diagrams are created by adult educators. In section 4.2, we mentioned that the orientation of the 2D solid-pyramid stimulus diagram might have contributed to an increased error rate on that item. The square base of the pyramid was shown to children at the bottom of the diagram depicting the solid pyramid. A future study might investigate the effects of the orientation of geometric objects in static diagrams to determine how producers of textbooks and educational materials might create the best learning tools for children to consider multiple aspects of geometric figures. We carefully placed target manipulatives in front of children so that differently-shaped faces of solid manipulatives were evenly exposed to children. By design, the orientation of figures in 2D diagrams did not mimic the orientations of the manipulatives presented so that some dynamic translation would be required to perform the task. We did not vary diagram or target orientations from one participant to another. Another possibly fruitful line of future inquiry is an investigation into the effects of the orientations of geometric figures in diagrams on children's spatial-mathematical reasoning.

5.1.3 Scaling Ratios

In section 4.3 we reported that children had trouble with the sorting task where they were required to scale shapes across large size differences. For the exclusion errors on the solid-rectangular prism- and solid cylinder-stimulus items, the size difference between stimulus and match was so great that children frequently were led to classify the same shapes as different. For the common inclusion error on the solid-cube stimulus item, the ratio of the sides of the small-rectangular prism was small enough that children seemed to automatically scale it into a match. Identifying the size difference at which children begin to routinely classify two identical shapes as distinct shapes would help educators teach children how to scale effectively. Similar studies that evaluate the influence of static-intrinsic ratios of same-class shapes on children's spatial-mathematical reasoning would also be helpful.

5.1.4 Pointiness

As presented in section 4.2, children quite often over-interpreted the significance of the points associated with triangles and triangular faces. Given the static-intrinsic characteristics of the triangular-prism manipulative, it is remarkable that no children were able to match the plane-rectangle stimulus and a rectangular face of the triangular prism. We also noted this phenomenon when children frequently misidentified the solid-cone manipulative as a match with the solid-pyramid stimulus item based on their mutual pointiness. This tendency seems especially open to intervention. Teachers might consider taking some time to discuss the common error when students are working with early geometry experiences. An effective intervention might be as simple as showing students a triangular prism, inviting them to make the error by replicating this item, then discussing how the plane rectangle actually matches to the triangular prism. Comparing a pyramid to a cone might also be helpful, emphasizing the difference between the bases. It would be a straightforward affair to investigate the effects of simple training interventions along these lines.

5.1.4 Dimensionality

In section 4.5 the issue of dimensionality was addressed. The volume of the solid-cylinder stimulus and the solid cone manipulative both gave children trouble in identifying how the plane rectangle and plane triangle fit the corresponding face of these items. Many adults are surprised when this principle is demonstrated, indicating it is not intuitively obvious. On several occasions children stated that voluminous manipulatives did not match projected 2D shapes because "it doesn't have sides." Since lines in a geometric diagram often represent the visual boundaries of real-world manipulatives, some adult scaffolding aimed at helping children understand this translation might help them envision more spatial relations than for children who do not receive this scaffolding. Another related question is, if children are shown various animation sequences of plane shapes projecting onto and out of corresponding solid shapes, might they exhibit improved spatial reasoning on composition and decomposition tasks? How might dynamic geometry programs be incorporated into e-text modules to facilitate these aims? The work by Ng and Sinclair (this issue) suggests that dynamic geometry interventions may increase children's awareness of how parts of diagrams relate to the whole shape being represented.

Additionally, when a voluminous solid-shape manipulative is held at eye-level and rotated on its horizontal axis, curved visual boundaries become straight visual boundaries at the proper degree of rotation. Training exercises that teach children these skills could also be investigated for their efficacy in helping children visualize a greater degree of spatial relations that exist.

6 Conclusion

This study found that children were able to effectively relate plane and solid shapes depicted in 2D diagrams to 3D manipulatives of those shapes. When children were asked to match the stimulus items to related component shapes, children in this sample exhibited some common difficulties. Firstly, their attention was unduly drawn to "pointiness," with triangular features absorbing inordinate amounts of feature significance. Secondly, children had difficulty imagining projected volume on 3D manipulatives collapsed down onto 2D, and they rarely aligned their visual perspective of shape manipulatives to help themselves overcome this challenge. Thirdly, children in this study erroneously excluded same-class shapes whenever the scaling difference between target and stimulus was greater than approximately 100% in size.

Duval's (1995) notion of operative apprehensions proved a useful tool for framing our assessment of children's reasoning about plane and solid shapes on this matching task. Concerning Duval's categories, the majority of error justifications given by children in this study related to failure to successfully make use of optic apprehensions. Imagining a shape whose boundaries are resized or whose dimensionality is altered but whose key class properties are retained proved challenging for these children. Neglected mereological apprehensions also occurred with regularity. Children had difficulty treating distinct faces separately in a consistent manner, such as in the case of the plane circle stimulus and the solid cone manipulative. Even though the circular base of the cone was often noticed as part of that shape, the pointy apex of the cone was too egregiously non-circular to ignore for a match.

We see potential for the use of Duval's framework in designing classroom curricula. We expect that spending time developing children's mereologic, optic, and place operations would increase children's visualization abilities when working with shape diagrams to develop spatial insights. We also agree with Duval's suggestion that avoiding cases involving complex operative apprehensions is probably counterproductive. Providing guided experiences with developmentally-appropriate, challenging geometric representations might bolster the visualization skills necessary to skillfully persist in mathematical development.

Additionally, this study found a similar trend in children's ability to explain their reasoning for classifying shapes as found in prior research (e.g., Clements et al., 1999). Namely, that the most common explanations given by young children on such tasks amounts to "I don't know." Young children are not confident in what they ought to be looking for when making shape-class judgments, elsewise they are overconfident in highly-salient features like points or large scaling differences between class examples. Thoughtful discussions soliciting children's attention to commonly-ignored features may help students overcome some of these natural tendencies toward error and increase their spatial vocabulary to help them express their reasoning. Future research investigating the efficacy of pedagogical interventions informed by this exploratory analysis would be especially welcome.

References

Clements, D. H., Swaminathan, S., Hannibal, M. A. Z., & Sarama, J. (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education, 30*(2), 192-212.

Duval, R. (1995). Geometrical pictures: Kinds of representation and specific processings. In *Exploiting mental imagery with computers in mathematics education* (pp. 142-157): Springer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1-2), 103-131.

Guay, R. B., & McDaniel, E. D. (1977). The relationship between mathematics achievement and spatial abilities among elementary school children. *Journal for Research in Mathematics Education*, 211-215.

Gunderson, E. A., Ramirez, G., Beilock, S. L., & Levine, S. C. (2012). The relation between spatial skill and early number knowledge: the role of the linear number line. *Dev Psychol, 48*(5), 1229.

Hegarty, M., Crookes, R. D., Dara-Abrams, D., & Shipley, T. F. (2010). Do all science disciplines rely on spatial abilities? Preliminary evidence from self-report questionnaires. In *Spatial Cognition VII* (pp. 85-94): Springer.

Herbst, P. (2004). Interactions with diagrams and the making of reasoned conjectures in geometry. ZDM, 36(5), 129-139.

Hiele, P. v. (1986). *Structure and insight: A theory of mathematics education*. New York.

IBM Corporation. (2013). IBM SPSS Statistics for Windows, Version 22.0. Armonk, NY: IBM Corp.

Lonergan, B. (1997). *Verbum: Word and Idea in Aquinas, Vol. 2 of the Collected Works of Bernard Lonergan*, F. Crowe, & R. Doran, (Eds.). Toronto: University of Toronto Press.

Newcombe, N. S., & Shipley, T. F. (2012). Thinking about spatial thinking: New typology, new assessments. *Studying visual and spatial reasoning for design creativity*. New York: Springer.

Ng, O., & Sinclair, N. (2015). Young children reasoning about symmetry in a dynamic geometry environment. *ZDM*, 47(3).

Noldus Information Technology (2008). The Observer XT 8.0 [Computer software], Wageningen: Noldus Information Technology.

Stannard, L., Wolfgang, C. H., Jones, I., & Phelps, P. (2001). A Longitudinal Study of the Predictive Relations Among Construction Play and Mathematical Achievement. *Early Child Development and Care, 167*(1), 115-125, doi:10.1080/0300443011670110.

Steenpaß, A., & Steinbring, H. (2013). Young students' subjective interpretations of mathematical diagrams: elements of the theoretical construct "frame-based interpreting competence". *ZDM, 46*(1), 3-14, doi:10.1007/s11858-013-0544-0.

Thom, J.S., & McGarvey, L. (2015). Living forth worlds through drawing: Children's geometric reasonings. *ZDM*, 47(3).

Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., et al. (2013). The malleability of spatial skills: A meta-analysis of training studies. *Psychological Bulletin, 139*(2), 352-402.

Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology, 101*(4), 817-835.

Willems, H. (1997). *Rahmen und Habitus: zum theoretischen und methodischen Ansatz Erving Goffmans; Vergleiche, Anschlüsse und Anwendungen*: Suhrkamp-Verlag.

Woods, T. A. (2009). Spatial Visualization and Imagery. In C. T. Cross, T. A. Woods, & H. Schweingruber (Eds.), *Mathematics Learning in Early Childhood: Paths toward Excellence and Equity* (pp. 183-196). National Academies Press.