SPATIAL REASONING FOR YOUNG LEARNERS

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This Research Forum proposal arises from a recent focus on spatial reasoning that began with a multi-year collaborative project amongst a diverse group of researchers (mathematicians, psychologists, mathematics educators) from Canada and the US, and continues to expand with the goal of: mapping out the terrain of established research on spatial reasoning; consolidating that research within a nuanced discussion of the actualities and possibilities of spatial reasoning in contemporary school mathematics; offering a critical analysis of the theories and practices that define contemporary curriculum and pedagogy of school mathematics in a range of countries and contexts, and, offer examples of classroom emphases and speculations on research needs that might help to bring a stronger spatial reasoning emphasis into school mathematics.

BACKGROUND

Currently little time is spent in early years classrooms focusing on geometry and spatial thinking (Uttal et al., 2012). In fact geometry and spatial reasoning receive the least attention of the mathematics strands in North America (Bruce, Moss & Ross, 2012; Clements & Sarama, 2011). However, there are several reasons to believe that this situation can and will change. The first is the extensive body of research over the past twenty years that has consistently shown the strong link between spatial abilities and success in math and science (Newcombe, 2010). Converging evidence from psychology research has revealed that people who perform well on measures of spatial ability also tend to perform well on measures of mathematics (Gathercole & Pickering, 2000) and are more likely to enter and succeed in STEM (science, technology, engineering and math) disciplines (Wai, Lubinski, & Benbow, 2009).

Second, there is a growing amount of evidence both from psychology and mathematics education showing that children come to school with a great deal of informal spatial reasoning (see Bryant, 2008), which is often not formally supported until much later in the curriculum, when numerical and algebraic ways of thinking have already become dominant. As early as four years old for example, children come to school with informal awareness of parallel relations, and although such relations are highly relevant to their work in two-dimensional shape identification and description, they are not formally studied until middle school. While such a concept might strike some educators as overly 'abstract' or 'formal', researchers have shown that given the appropriate and engaging learning environments, k-2 children can, indeed, develop very robust understandings of parallel lines (see Sinclair, de Freitas and Ferrara, 2013).

This leads to the third reason for change, which concerns the increase of digital technologies for young learners. In contrast to older software, such as Logo-based programming, which require numerical and/or symbolic input, or older mouse- and keyboard-drive hardware input, which can present motor dexterity challenges, newer touchscreen and multi-touch environments can greatly facilitate mathematical expression (Bruce, McPherson, Sabbeti & Flynn, 2011). Research has already shown how new digital technologies that promote visual and kinetic interactions can help support the teaching and learning of spatial reasoning (Clements & Sarama, 2011; Highfield & Mulligan, 2007; Sinclair & Moss, 2012). These new technologies are challenging assumptions about what can be learned at the k-5 level; they are also showing that long-assumed learning trajectories might change drastically if spatial reasoning becomes a more central and explicit component of the curriculum.

Each of the sessions includes a plenary opening presentation that provides background on the historical, epistemological, psychological, mathematical and curricular contexts of spatial reasoning. Following each plenary, a coherent grouping of poster presentations illustrating classroom-based empirical studies will be used to incite small group discussions. Each round of posters will be followed by a facilitated whole group discussion (30 minutes) that highlights common themes and possible contradictions from the small group discussions and demonstrations. The first session critically analyses the role of spatial reasoning in the mathematics curriculum. The second session focuses on the role of different technologies in 'spatialising' the mathematics curriculum.

SESSION 1. THE ROLE OF SPATIAL REASONING IN MATHEMATICS LEARNING

This section contains two papers, followed by a five poster presentation descriptions. These will be used as a basis to motivate discussion around the following key questions:

1. What other factors may be contributing to the low emphasis on spatial reasoning in the curriculum? How might spatial reasoning fit into the numeracy strategies that many countries are pursuing?
2. What other strategies can be used to "spatialise" the curriculum and what impact might these have on assessment and professional development?

Where mathematics curriculum comes from

Brent Davis (University of Calgary)

The history of school mathematics is so tangled that attempts to offer a fulsome accounts necessarily drift toward fiction. The intention here is thus more to hint at its complexity.

With all the resources devoted to curriculum development, one would be justified in thinking that school mathematics undergoes constant reinvention – but curricular similarities across nations and centuries suggest otherwise. In other words, most curriculum developers don't actually develop curriculum. Their roles are more toward engineering programs of study.

How, then, did school mathematics come to be a trek from arithmetic through algebra to calculus, with modest diversions into geometry, statistics, and other amusements? To answer, I focus on three moments in its history, using a handful of personalities as metonyms for clusters of developments. To re-emphasize, the aim is neither completeness nor accuracy. For those I defer to others (Bishop et al., 1996; Howson, 1973; Menghini et al., 2008; Schubring, n.d.; Stanic & Kilpatrick, 2003).

Moment 1– up to the mid-1600s

The Pythagoreans (c. 500 BCE) are often credited with many innovations to schooling, both structural and curricular, but their major contributions are conceptual and philosophical. They helped to collect a rather disjoint set of facts and insights into a coherent, powerful system of knowledge that would later be at the heart of Plato's liberal arts – arts that are freeing.

However, "mathematics" was not part of those liberal arts, simply because it didn't exist as a coherent disciplinary domain. The devices were not yet invented to unite Logic, Arithmetic, Geometry, and other domains, but Euclid (c. 300 BCE) took a major step in that direction as he employed logic to prove, connect, and extend geometric truths. This massive intellectual leap set the stage for a unified discipline … but that unification had to wait almost two millennia until the 1600s when René Descartes brought together arithmetic, analysis, geometry, and logic through the masterstroke of using a coordinate system to link number and shape. That extraordinary contribution marked the emergence of the system of knowledge that we know as mathematics, affording a means to gather not just notions of number, shape, and argument, but also a host of other foci now recognized as properly mathematical.

However, Descartes did very little to shape school mathematics. Major contributions in that regard actually predate him – and that fact is telling. The content and foci of what was to become school mathematics were largely defined before mathematics cohered as a domain of inquiry, as exemplified in Robert Recorde's work. His 1540 textbook emphasized notations, procedures, and applications.

Moment 2 – mid-1600s to mid-1990s

There have actually been two distinct school mathematics through most of modern history, only merging in the mid-1900s within a broader education-for-all movement.

In the elementary schools of the newly industrialized world, curriculum was more aligned with the work of Recorde than that of Descartes. It was about ensuring that the workforce would have a functional numeracy.

Secondary school mathematics was quite a different beast. Intended more for elites, there mathematics more reflected what Descartes had brought forth. However, as the two projects converged, its contents were increasingly selected and framed through the utilitarian mindset of elementary schools.

Moment 3 – from the mid-1900s

Even so, the merging of elementary and secondary school mathematics was never made seamless. Teachers have experienced strategies aimed at unification as pendulum swings between rote competence and deep understanding; students have experienced them as discontinuities, reflected in the too-frequent confession of being "good at math until Grade 6."

Even so, revisions since the mid-1900s have proceeded as though school math were unified. Triggered by a sequence technological and economic rivalries with the USSR, Japan, and China, the progression of New Math, Reform Math, and the New New Math represented efforts to force coherence onto topics that were never chosen for that purpose. Consequently, while pedagogical emphases changed, the substantive content remained stable.

What's holding current curriculum in place?

With that stability, we force children to master competencies that are increasingly (if not completely) irrelevant as we ignore topics of growing necessity. Ironically, mathematics curriculum appears to find its stability in the conflicting interests of stakeholders. Facets of these include a culture of examination, a profit-driven textbook industry, inflexible university mathematics departments, and a self-perpetuating cycle of teachers teaching as they were taught – coupled to the fact that curriculum is a result, not an input. It cannot serve as a mechanism to effect change; it can only co-evolve with shifts in belief and expectation.

How is spatial reasoning related to mathematical thinking and how important is early exposure to spatial activities?

Yukari Okamoto (University of California Santa Barbara); Lisa M. Weckbacher (California State University Channel Islands); David Hallowell (University of California Santa Barbara)

There is an urgent need to foster students' spatial reasoning skills. This need is recognized in the recent NCTM Focal Points that places its emphasis on geometry as one of the three key curriculum focal points (NCTM, 2006). At the same time, empirical evidence is emerging that suggests that those with a strong sense of space tend to be successful in mathematics and more broadly in the STEM disciplines (e.g., Newcombe, 2010; Wai, Lubinski, & Benbow, 2009). One goal of this presentation is to synthesize the recent findings that support the link between spatial reasoning and performance in mathematics and other STEM disciplines. Another goal is to discuss studies addressing the link between early spatial experience and later spatial sense. Below I will briefly present the studies conducted in my lab.

How is spatial reasoning related to mathematical thinking?

In one study, we asked 114 high school students (56 females) to fill out various measures of spatial thinking as well as to take a geometry test that consisted of age appropriate items taken from publicly available sources such as National Assessment of Educational Progress (NAEP, 2006). The goal was to examine whether we find differences in geometry performance between those who scored high on spatial measures and those who scored low on such measures. Because spatial cognition is a multi-faceted construct, we used three existing measures that assess different aspects of spatial thinking: Mental Rotation Task (MRT; Vanderberg & Kuse, 1978); Paper Folding (Ekstrom, French, Hartman, 1976); and Snowy Pictures (Ekstrom, French, & Harman (1976).

Using the composite scores, we identified 23 participants to be strong in spatial reasoning (high spatial) and 25 participants to weak in such reasoning (low spatial). There was a large, significant difference between the two groups on the test of geometry. In addition, high spatial participants had higher grades in geometry. They also earned higher grades in algebra but this difference did not reach statistical significance. When the geometry test items were separated into either 2- or 3-dimensional items, the high spatial group did significantly better on the 2-dimensional items than the other group and the difference on the 3-dimensional items approached significance in favor of the high spatial group.

These findings demonstrate a strong link between spatial sense and mathematical performance (at least in geometry). Lack of statistical differences in algebra grades and 3-dimensional items are not necessarily discouraging. After all, these high school students were taught mathematics that did not emphasize geometry or spatial solutions to algebraic problems. Had geometry and spatial thinking been emphasized, spatially oriented students would have excelled not only in 2-dimensional geometry but also in ways to approach complex situations spatially.

How do first graders understand geometric shapes?

We are currently conducting a study to understand how first-grade children reasoning about plane and solid shapes. We are particularly interested in how children interpret various geometric shapes when asked to compose or decompose them.

Our initial effort includes 15 first graders. We devised four composition and four decomposition items. For each item, there was a stimulus item and four option shapes. One option shape was exactly the same as the stimulus shape; two others matched but included distracting features; and one item was non-match. For each of the composition items, children saw a stimulus item that is a 2D diagram of a plane geometric figure. They were then shown four other geometric figures that are solid or plane 3D figures. Their task was to choose among the four options the figures in which the stimulus shape is contained. That is, the stimulus shape is a composite of the selected figure. For each of the decomposition items, children saw a stimulus item that is a 2D diagram of a solid geometric figure. Similar to the composition items, they were then shown four solid or plane 3D figures. Their task was to choose among the four options whose figures consist of all or part of the stimulus shape. That is, the stimulus shape that is solid must be decomposed to 2D components that make that solid shape.

Our preliminary findings indicate that first graders were able to find option shapes that were exact matches to the stimulus shapes. But when some distracting features were included, they had difficulty recognizing matched features in either composition or decomposition items. Interestingly, children found triangular vertices ("pointiness") as a significant feature to accept or reject option figures. Overall, we found inconsistency in children's criteria for choosing features of geometric shapes in deciding a match or non-match.

How important is early exposure to spatial activities?

Our interest in this line of work is to examine types of activities in which highly spatial individuals have participated throughout their lives. We are particularly interested in activities in which they participated early and those in which they continue(d) to participate for a prolonged time period. We are currently collecting data from junior high, high school and college students. Here, we describe our first effort that investigated gifted 7th and 8th graders.

We identified 14 students (9 females) as highly gifted in quantitative and/or verbal reasoning. They all filled out a survey that listed 70 spatially oriented activities in five different areas: computers, toys, sports, music and art. The group as a whole reported that their favorite activities included video games, blocks, board games, soccer, piano, and drawing. Because our participants included students who were identified as gifted in quantitative and/or verbal reasoning, we decided to give two types of spatial measures to further identify those who are strong in mental rotation abilities (MRT) and those who use verbal cues to process spatial information (verbal-spatial task (VST); adapted from Hermelin and O'Connor, 1986). This resulted in five students who scored high on MRT but low on VST and five students who were high on VST but low on MRT. We found that high MRT/low VST students favoured activities similar to the rest of the students in the study but for a longer period of time. These students consistently favored these activities from early childhood to the present time. On the other hand, there were no consistent patterns that emerged in the favored activities by the high VST/low MRT students.

These findings show that gifted students in general and those strong in mental rotation in particular tend to favor activities that include highly spatial elements and the latter group in particular engaged in such activities for a long period of time. This study suggests there is a link between spatial activities and spatial thinking (mental rotation in this study). As we collect more data from a wider age range of students, both gifted and non-gifted, as well as use measures that assess different aspects of spatial reasoning, we hope to more clearly explain how early spatial experiences contribute to later spatial and geometric reasoning skills.

POSTERS: (RE) 'SPATIALISING' THE CURRICULUM

Changing perceptions of young children's geometry and spatial reasoning competencies: lessons from the "Math for Young Children" (M4YC) Project

Joan Moss, Zack Hawes, Sarah Naqvi, Beverly Caswell (OISE)
Catherine D. Bruce, Tara Flynn (Trent University)

We are witnessing an unprecedented political and academic focus on mathematics in early years classrooms which has included a call for a greater emphasis on geometry and spatial reasoning. Spatial reasoning in early years is foundational, not only to later success in mathematics (Mix & Cheng, 2012), but also to success in the STEM disciplines (Wai, Lubinski, & Benbow, 2009; Newcombe, 2010). Unfortunately, not all children have equal access and exposure to spatial reasoning (e.g., Casey, et al., 2008): indeed, recent research has shown striking SES-related differences in spatial reasoning in children as early as 3-years of age (Verdine, et al 2013).

The Math for Young Children Project

For the last 3 years, we have been working on a professional development research project to promote and enhance the teaching and learning of geometry and spatial reasoning in early years classrooms. Using a Japanese Lesson Study approach, the Math for Young Children project has, to date, collaborated with more than 15 teacher-researcher teams which have included more than 100 early years teachers (pre-school to second grade) and their students. Demographically, we have been working in underserved populations, typically in schools with low provincial test scores.

Our work with our teacher-researcher teams has been the co-design and implementation of lessons, activities, assessment tools and trajectories that build on the development of children's geometrical and spatial reasoning. An important mission of this project has been to gather data to demonstrate that young children – regardless of SES background – are capable of exceeding current expectations in geometry and spatial reasoning given carefully crafted learning experiences.

In this poster we focus on the work of one team of 8 teachers and their Kindergarten and Grade 1 students from a large urban low-SES school. We present the design, implementation and results of two “Research Lessons”, both of which involved knowledge and geometric reasoning well beyond curriculum expectations.

The first lesson, conducted with 3-5 year olds, centered on the “pentomino challenge” involved students’ in discovering the twelve unique shapes, composed with 5 squares (see figures 1 and 2).

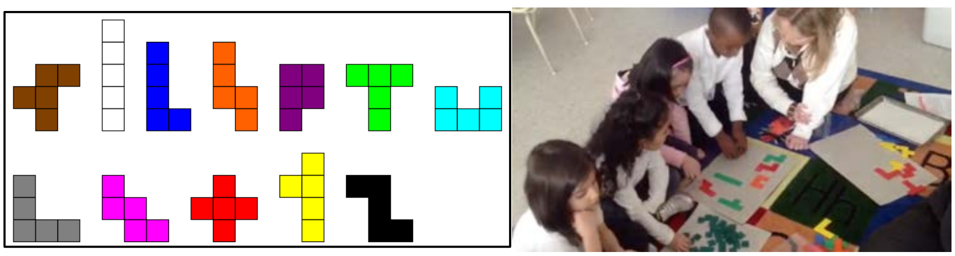


Figure 1: Set of 12 pentominoes

In the second lesson, “The Upside Down World, Grade 1 students were challenged to recreate ‘buildings’ composed of multilink cubes in their upright orientation and use spatial language to describe the composition of the ‘buildings for other class members to build accordingly (see figure 2).

Figure 2. Grade 1 students working with teacher on Upside-down World lesson

Results

Overall the results revealed that the majority of the 4-, 5- and 6-year-old students performed well above expected levels on activities involving aspects of geometry typically reserved for older students in later grades. Specifically, among our findings were that the kindergarten students were able to recognize, rotations and reflections in 2-dimensional figures, demonstrate an understanding of congruence and, collectively, find all twelve pentomino configurations (see figure 2). The grade 1 students showed the ability to: 1) copy and assemble 3-dimensional shapes composed of multi-link cubes; 2) discriminate between congruent and non-congruent 3-dimensional figures; and, 3) describe to classmates the necessary steps required to rebuild a shape that was flipped upside down from its intended orientation. Moreover, students as young as kindergarten age demonstrated a great interest in the activities and were able to sustained their interest and engagement in the tasks and lessons.

The application of ambiguous figures to mathematics: in search of the spatial components of number

Michelle Drefs & Lissa D'Amour (University of Calgary)

Ambiguous figures allow for a single image to be correctly perceived and interpreted in more than one way. A classic example is the "My Wife and My Mother-in-law" illustration, in which the image can be viewed as either the side profile of a young woman or as an old woman from the front (Boring, 1930). Which image first emerges depends on what information the viewer attends to and privileges within the display (e.g., de Gardelle, Sackur, & Kouider, 2009). The same principles can be seen to apply when examining the tasks and activities used to develop students' understanding of number and number systems. Specifically, many commonly used number tasks are viewed as predominately, if not completely, numerical in nature. However, evidence is accumulating in support of mathematical knowledge as having a strong spatial component (e.g., Bishop, 2008). In fact, both domain-specific (e.g., Approximate Number System; Butterworth, 1999) and domain-general (e.g., visuo-spatial processing) spatial mechanisms are believed to undergird numerical competencies and mathematical thinking (see Bonny & Lourenco, 2012; Sasanguie, Göbel, Moll, Smets, & Reynvoet, 2013). In alignment with this, several researchers have identified spatial aspects as involved in a number of commonly utilized numeracy tasks and activities. For example, LeFevre and collaborators (2010) identified that spatial attention plays an important role in preschool and kindergarten children's ability to successfully complete number naming, numeration, and symbolic magnitude comparison tasks as well as to areas of mathematic achievement two years later. Ansari et al. (2003) similarly found visuo-spatial abilities to account for typically-developing children's success on standard "How many?" and "Give a number" cardinality tasks. Aside from directly impacting student's number knowledge, spatial cognition may additionally contribute indirectly with, for example, spatial visualization supporting word problem-solving performance (e.g., van Garderen, 2013). The importance of spatial cognition to number knowledge is also highlighted in the work of Mowat and Davis (2010) in which they discuss the importance of utilizing sensori-motor experiences to deeply understanding a particular math concept. They argue that inherent to an understanding of number as a "position in space" is the spatial component of movement along a continuous measurement scale.

To identify aspects of “spatial cognition” of relevance and importance to mathematical tasks and activities requires, as with ambiguous figures, a shift in what information is being attended to and privileged. The purpose of this poster session is thus to explore this “ambiguous” interplay between number and spatial cognition as it relates to number as a focal point in the curriculum. Specifically, it is intended that participants will be provided an opportunity is to explore the possible intersects between number and spatial cognition. Emphasis will be given to: (1) outlining several cognitive mechanisms that provide support for a spatial component of number, and (2) providing common number tasks and activities that attendees can examine and discuss with respect to the contributions of both spatial and number knowledge.

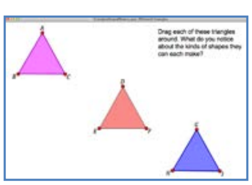
Young children’s thinking about different types of dynamic triangles

Harpreet Kaur (Simon Fraser University)

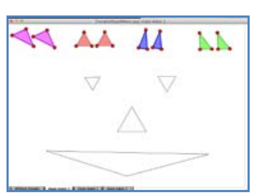
The aim of this study is to show how young children (age 7-8, grade 2/3) can exploit the potential of dynamic geometry environments to identify, classify and define different classes of triangles (scalene, isosceles, equilateral). Our motivation was to explore what can be further done on the topic of triangles in the lower primary school while remaining within the topic of the curriculum but extending the geometric applicability and sophistication.

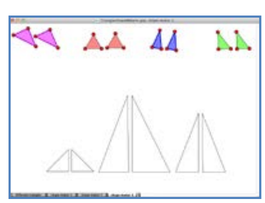
We developed the triangle Shape Makers sketches (see some examples in figure 1 a, b, c, d) for different types of triangles (scalene, isosceles, equilateral triangles, right triangle) to extend the work of Battista (2008). Each triangle type had a different colour (pink for scalene, red for equilateral, blue for isosceles and green for right). In the sketch shown in figure (1a) only the middle triangle is constructed to be equilateral. Sketches in figure 1 (b, c, d) were used as a way of focusing attention on the inclusive relations.

(a) Drag each of these triangles around. What do you notice about the kind of shapes they can each make?



(b) & (c): Which coloured triangles can fit into given triangle outlines?





(d) Whether a scalene triangle can fit into the given equilateral triangle outline (top) and vice versa (bottom)?

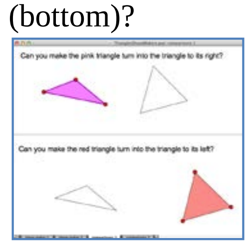


Figure 1 (a, b, c, d): Different triangle sketches

The research was undertaken in the context of a classroom-based intervention. Three lessons on triangles were conducted with the 24 children seated on a carpet in front of an interactive whiteboard. Previous lessons with *Sketchpad* involved the concepts of symmetry and angles, but the children had never received formal instruction about classification of triangles before. We used Sfard's (2008) communicational approach for looking at children's discourse (thinking) about dynamic triangles along with their gestures, use of diagrams during their interactions with dynamic sketches.

(a) (b) (c)

Figure 2 (a, b, c): Snapshots of various gestures by children

Over the course of their interaction with the dynamic triangles, children's discourse started with noticing the informal properties based on dragging behaviour and eventually moved to formal properties (e.g. angles are staying same in the equilateral triangle). The children and the teacher used "if…then" statements extensively to talk about the behaviour of the different triangles in the dynamic environment. The dynamic behaviour of the triangles prompted the children to make connections of the restricted or free movements of the dynamic triangles with real life experiences of the restrictive mobility of humans (e.g. children used word 'paralysed' for isosceles triangles). Children made ample use of gestures (see figure 2 a, b, c) during the intervention (e.g. stretching both arms to show moving behaviour of sides of isosceles triangle, gesturing a triangle with the fingers of both hands, tilting head sideways to recognise non-prototypical isosceles triangle,). The children's communication shifted from specific to more generalized statements as intervention progressed, which included inclusive descriptions of classes of triangles (in other words, they thought about equilateral triangles as special types of isosceles triangles). This kind of reasoning emerged as a result of the children's attempts to overlap or fit different triangles on each other in Sketchpad. This study also provides the initial evidence that the teaching of concepts like symmetry and angles in early years can lead to whole set of new possibilities of geometric reasoning about shape and space for young children.

Kindergartners' abilities in perspective taking

Marja van den Heuvel-Panhuizen (Utrecht University, the Netherlands) and Iliada Elia (University of Cyprus) and Alexander Robitzsch (Federal Institute for Education Research)

Contemporary early childhood curricula and educational programs emphasize the need to start with 3D geometry at an early age (NCTM, 2008; Van den Heuvel-Panhuizen & Buys, 2008). This approach is in line with Freudenthal's (1973) view regarding early geometry learning: "Geometry is grasping space […] that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it" (p. 403)

Thus, spatial ability is important for young children to learn and therefore, it is worthwhile to gain more insight into how they develop this ability. A major component of spatial ability is the competence of imagining objects from different perspectives of the viewer (e.g., Hegarty & Waller, 2004). In the present study, we focused on this specific spatial competence of kindergartners, namely, imaginary perspective taking competence (IPT), which means that children are able to mentally take a particular point of view and that they can reason from this imagined perspective.

Flavell, Everett, Croft, and Flavell (1981) proposed and validated a distinction into two abilities of perspective taking. The Level 1 competence concerns the *visibility of objects,* that is, the ability to infer which objects are and are not visible from a particular viewpoint. The Level 2 competence is related to the *appearance of objects, that is, the ability* to make judgments about how an object looks from a particular viewpoint.

The aim of this study was to gain more insight into kindergartners' IPT and specifically IPT type 1 (visibility) and IPT type 2 (appearance). Also, we intended to identify cross-cultural patterns in this competence and therefore we included children from two countries in our study. In particular, we investigated how able kindergartners are in IPT type 1 and type 2, how these competences are related, and whether the IPT competence is related to children's kindergarten year, mathematics ability and gender. Furthermore, we examined whether there are cultural similarities and differences in these IPT competence issues.

The sample consisted of 4- and 5-year-old kindergartners in the Netherlands (*N*=334) and in Cyprus (*N*=304). Children's IPT competence was assessed by a paper-and-pencil test of various perspective-taking pictorial items which require either IPT type 1 or IPT type 2. In Figure 1 two test items are given. The *Duck* item (instruction: "The duck has fallen into the hole. He looks up. What does he see?") is meant for measuring IPT type 1, while the *Soccer* item (instruction: "Two children are playing soccer. How do you see it if you look from above like a bird?") is meant for measuring IPT type 2.

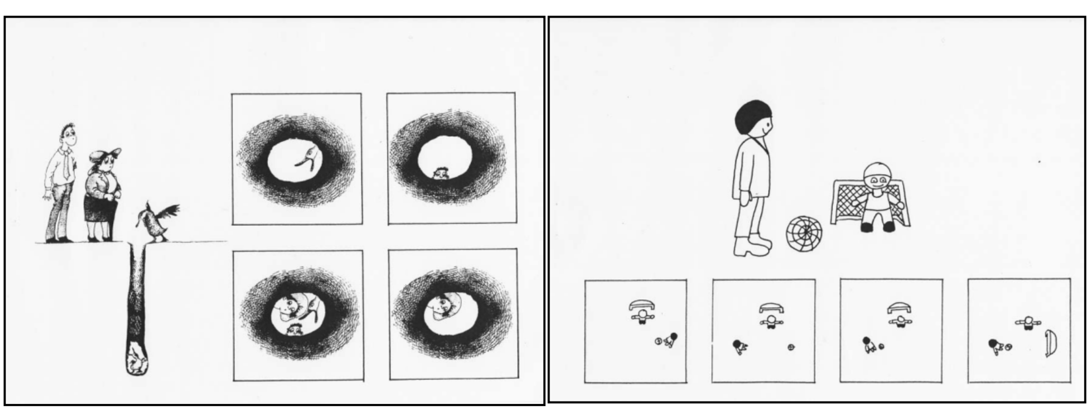


Figure 1a: *Duck* item   Figure 1b: *Soccer* item

The results revealed interesting common patterns for the two IPT types in both countries. Specifically, IPT 2 items were significantly more difficult than IPT 1 items and children's success on the former items implies success on the latter items. Also in both countries, IPT 1 appeared to develop during the kindergarten years. For IPT 2 this was the case only in the Netherlands. In the two countries, there were no significant gender differences for kindergartners' IPT competence. However, the relationship between children's IPT competence and mathematics ability was not so clear, as in the Netherlands and in Cyprus significant interaction effects were found.

A spatial-visual approach to optimization and rates of change

Robyn Ruttenberg (York), Ami Mamolo (UOIT), Walter Whiteley (York)

Optimization and rates of change are familiar terms to calculus students, but what do these concepts look like when lifted from that context and interpreted through a spatial-visual lens? We present research on elementary students' spatial-visual with a well-known optimization problem: the "popcorn box problem." We will illustrate how different tools and representations, and their affordances– in our case 3-D models – can provide enriched learning experiences for pupils by allowing them entry and access to sophisticated mathematics without the need for computation or calculation.

The popcorn box problem:

> Given a square sheet of material, cut equal squares from the corners and fold up the sides to make an open-top box. How large should the square cut-outs be to make the box contain maximum volume?

To investigate this problem, we introduced a network of activities which culminated in the physical exploration of pairs of clear-plastic boxes with coloured foam inserts that represent volume lost and gained when comparing two boxes (see Figure 1, below). Our poster highlights key features of the activities, as well as our design considerations for scaffolding the exploration. Via engagement with the activities and models, several mathematical ideas and observations emerged in a manner accessible to learners of various ages and mathematical sophistication. We identify a few key ones accessible to young pupils here:

Figure 1: Pairs of clear plastic 'popcorn boxes'

(i) The volume of the boxes can change, and boxes with different shapes could have the same volume.
(ii) There is a largest volume.
(iii) The maximum volume is *not*: at either extreme, in the cube shape, or the 'middle' between extremes.
(iv) Volume and surface area can be physically represented, and physically compared in multiple consistent ways.
(v) Change in volume between pairs of boxes (Figure 1) involves both volume lost (on the sides of the outside box) and volume gained (on the top of the inside box), as the cut size is increased.
(vi) Given the uniform thickness of these gains and losses (the size of the increase in the cut), these changes in volume (loss and gain) between pairs of boxes can be compared with clarity by naïve overlay strategies (see Figure 2).
(vii) This reasoning can be extended to other pairs of boxes by adapting approximate loss-gain representations.
(viii) That the side ratios of the optimal shape were invariant under scaling (proportional reasoning).

We present some of the challenges and coping strategies that emerged as pupils engaged with the activities. Challenges such as how to accurately compare two boxes "close" in volume (but possibly very different in shape), and how to compensate for the physical constraints of imperfections of the tools, led to the negotiation of new strategies for comparing boxes as well as discussions of how to refine comparisons. Results suggest that big ideas about volume, and changes in volume, that students at the Grade 9 level had not noticed with regular 'volume related' calculations in their curriculum, are accessible to young pupils through our approach.

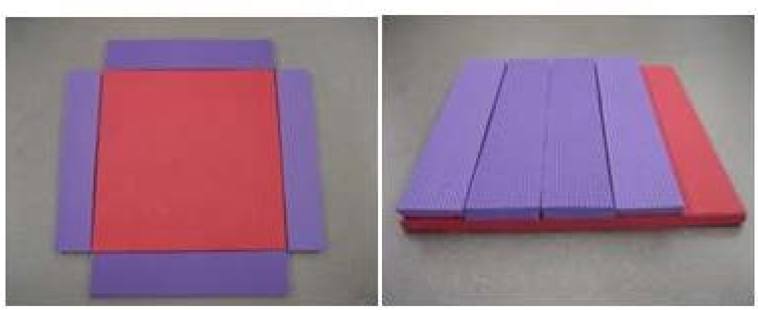


Figure 2: Foam inserts removed and compared via overlay

SESSION 2. SPATIALIZING THE CURRICULUM: CLASSROOM TOOLS AND TECHNOLOGY IMPLICATIONS

The second part contains two papers and five descriptions of poster presentations. These will motivate discussion on the following key questions:

1. What role do different tools (both digital and non-digital) play in promoting spatial reasoning in learning mathematics?
2. How might spatial reasoning be linked to the kinds of kinaesthetic and haptic forms of interaction that are available in new multi-touch platforms?

The malleability of spatial reasoning and its relationship to growth in competence

Diane Tepylo, Joan Moss & Zack Hawes (OISE)

It what ways can we support the development of young children's mathematical thinking; ways that are equitable to all young children? In this presentation we highlight the potential role of spatial reasoning and argue for the importance of a "spatial education" for young children.

Over a century of research confirms the close connection between spatial thinking and mathematical performance (Mix & Cheng, 2012). Recently, researchers have found that spatial thinking is not only correlated with mathematics but also predictive of later mathematics performance. To investigate the influence of early spatial learning skills on later mathematics, Verdine and colleagues (2013) carried out a longitudinal study with 3- to 5-year olds, in which they assessed children's performance on standard vocabulary and mathematics measures, as well as on measures of spatial reasoning in the form spatial assembly tasks. A surprising finding was that children's spatial reasoning skills at age three were the strongest predictor of mathematical skills at age five—even more than the three-year olds' math skills (Farmer et al., 2013). Another surprising finding was the significant SES-related difference in the spatial skills of three-year-olds.

In another longitudinal study, researchers investigated how the quality of kindergarten students' block play related to their performance in mathematics up to ten years later. Remarkably, kindergarten students' block building skills predicted their mathematics success in high school, even after controlling for IQ (Wolfgang et al). Furthermore, brain imaging studies corroborated the link between spatial and mathematical processing, revealing that overlapping brain regions are active during the performance of both spatial and of mathematical tasks. Given the proven relationship between spatial thinking and mathematics, an important question emerges: Is it possible to improve spatial reasoning skills?

Once believed to be a fixed trait (e.g. Newcombe 2010), there is emerging evidence that spatial thinking is malleable. A recent meta-analysis of 217 training studies surveying more than 20 years of research, concluded that spatial thinking can be improved in people of all ages through diverse sets of training activities. The implications are far reaching, especially with respect to early interventions which have been repeatedly shown to be most effective in bringing about long-term change (Heckman, 2006). Many spatial training efforts carried out with young children demonstrate that it is possible to improve the children's spatial skills, reduce or eliminate early gender differences and, critically, reduce differences attributable to SES.

These findings lead to the next question: can training in spatial reasoning support mathematics performance? This is a long-standing issue in the cognitive science field: making a causal link between spatial training and math performance has proved to be difficult. Recently, however, two studies clearly link spatial training and improved math ability: the first study, with at risk children in a long term afterschool program Grissmer et al., (2013), and the second study with typically developing children, in a single 20 minute training program (Cheng and Mix. 2013).

In a controlled random assignment afterschool intervention study, Grissmer and colleagues (2013) invited kindergarten and Grade 1 children to construct and copy designs made from a variety of materials including Legos®, Wikki Stix®, and pattern blocks. A control group was given a non-spatial curriculum. After 7 months, 4 days a week, the children in the experimental group made substantial gains in their mathematics and spatial reasoning, moving from the 30th to the 47th percentile on a nationwide test of numeracy and applied problems. Most striking, there was no "mathematics" taught as part of the intervention, nor did the instructors ever specify any connections between the construction activities and mathematics. There were no gains in mathematics performance in the control group.

In the second intervention, Cheng and Mix (2013) randomly assigned 6 and 7 year old students to either a single-session mental rotation-training group or a crossword puzzle group. Children in the spatial training group, but not the crossword group, demonstrated significant improvements on their abilities to solve problems involving place value and addition and subtraction. Surprisingly, the greatest improvement was on the difficult missing terms problems such as 5+ __ = 11. The researchers posit that the spatial intervention may have encouraged children to mentally rotate the question into the more common equation: 11 – 5 = __.

There is growing awareness of the foundational role spatial reasoning has in mathematics and many scientific disciplines, but, notably, spatial reasoning is rarely taught. The NRC expresses concern in their 2006 spatial reasoning review: "spatial reasoning is not only under supported, under appreciated, and under valued, but it is underinsturcted" (p. 5). We join them in their commitment to the development of spatial thinking across the curriculum.

The role of tools and technologies in increasing the types and nature of spatial reasoning tasks in the classroom

Cathy Bruce (Trent University) and Nathalie Sinclair (Simon Fraser University)

Since the inception of Kindergarten, mathematics learning tools have been envisaged as a feature of most mathematics classrooms for young children. For example, Friedrich Froebel, the progenitor of Kindergarten, and Maria Montessori, were powerful influences in the first policies related to Kindergarten, proposing that young children could engage in serious intellectual mathematical work; and both developed specific mathematical tools for students to use in order to explore these mathematics concepts. In fact Froebel's first ten sets of thinking tools, known as 'gifts' for the students, were 3-D figures that encouraged children to explore spatial reasoning and geometry (http://www.froebelgifts.com/). More recently, digital learning tools have been introduced to learners, the first of which relied on mediating objects such as a mouse, joystick or keyboard to operate the technology.

Over the past decade, several researchers have argued for the appropriateness and benefit of using "virtual manipulatives" (VMs) in the early grades, which build on the familiarity of physical ones, but which may also provide a range of added affordances. These researchers have questioned the assumption that "concrete" tools are more appropriate for young children and have argued that physical manipulatives are limited in their ability to promote both mathematical actions and reflections on these actions (Sarama & Clements, 2009). These authors point specifically to a VMs potential for supporting the development of *integrated-concrete* knowledge, which interconnects knowledge of physical objects, actions on these objects and symbolic representations of these objects and actions. Beyond VMs, there are also digital technologies that have little or no relation to physical manipulatives, but that also have unique potential for younger learners—especially dynamic mathematical technologies that help focus attention on mathematical relations and invariance as well as providing rich, visual examples spaces for mathematical objects such as triangles and numbers (Battista 2007; Bruce et al. 2011; Highfield & Mulligan 2007; Sinclair & Crespo, 2006; Sinclair, de Freitas & Ferrara 2013; Sinclair & Moss 2012).

With the advent of interactive whiteboards, touchscreen technologies were introduced to classrooms, and the role of technology in learning instantly expanded well beyond the initial observed roles. In an early publication on touchscreen technologies, Pratt and Davison (2003) describe the "visual and kinesthetic affordances" of the interactive whiteboard. Visual affordances relate to "the size, clarity and colourful impact of the computer graphics, writ large on the whiteboard" (p. 31). Kinesthetic affordances relate to "the potential impact of dynamically manipulating the screen in such a way that the teacher's (or child's) agency in the process is far more impressive than merely following a small mouse arrow" (p. 31). With further advances of touchscreen technologies, including tablets and handheld devices, direct-touch response has once again sparked new directions in educational research. Researchers are observing that in comparison to handling physical tools, engaging with virtual tools on touchscreens provides students with *different* kinesthetic experiences. The technological-pedagogical interactivity (see figure 1) which is engendered in a technology-mediated learning environment (Bruce, Flynn, 2012) is particularly salient in relation to spatial reasoning.

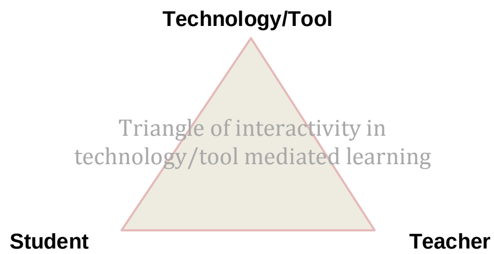


Figure 1. One conception of the technological-pedagogical interactivity triangle in technology mediated learning environments (Bruce & Flynn, 2012)

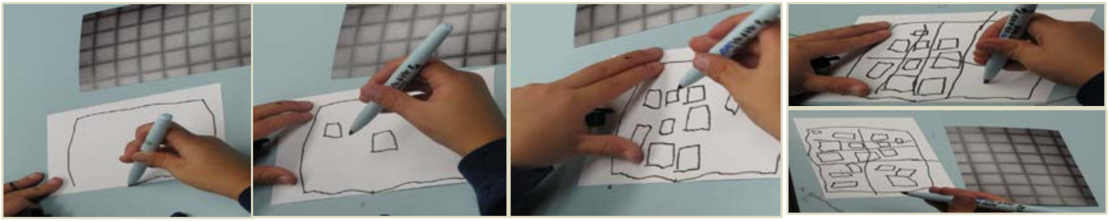
In an alternative approach to tool-use, de Freitas and Sinclair (2014) view the student, teacher, tool, mathematical concept as an assemblage that intra-acts because the distinctions between each "body" is not necessarily pre-determined. The focus thus shifts from the student or the tool to the interactive student-tool relation. In terms of spatial reasoning, they show how particular finger movements on the screen enable new ways of seeing numbers and operations. Gestures are one of the significant forms of finger movements used on touchscreen technologies and recent research that draws on the role of gestures in thinking and learning mathematics more broadly has pointed to their potentially unique role within the interactive environment of touchscreen technologies (Bruce et al. 2011; Sinclair, de Freitas & Ferrara 2013) and number (Sinclair & Heyd-Metzuyanim, 2014; Sinclair & Pimm, 2014).

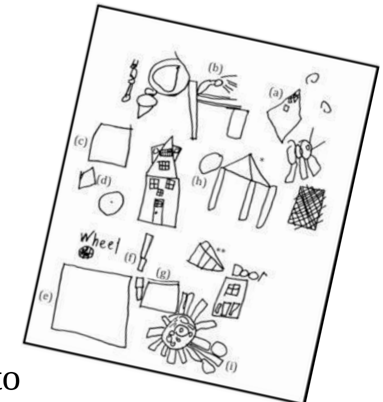
Recent technologies and tools are also putting into question the typical estimations of what children can do and understand spatially. Consider for example, how iPad applications such as *TouchCounts* (Sinclair & Jackiw, 2011) and *Spot the Dots* (Bruce) encourage young children to use spatial reasoning and gesture to cement foundational number concepts of quantity, magnitude, ordinality, cardinality and composition of number. In a unique tool design study, Hawes et al. (in press) have developed physical materials that are proving how children as young as 4.5 years old can imagine complex rotations of 3-D figures – a skill previously demonstrated to be too difficult for this age.

POSTERS: THE ROLES OF TOOLS IN SPATIAL REASONING

Children's drawings: A bodying-forth of spatial reasonings

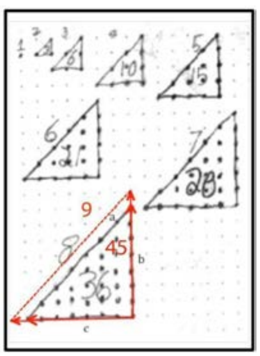
Jennifer S. Thom (University of Victoria) & Lynn M. McGarvey (University of Alberta)

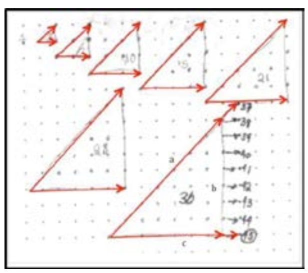


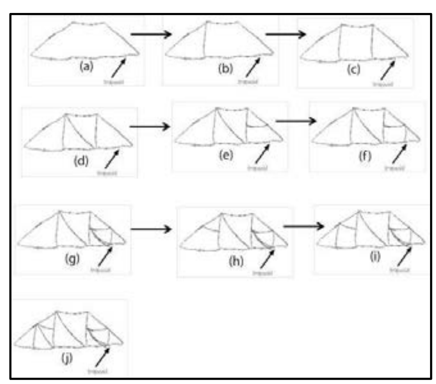


There is a growing interest in mathematics education to better understand children's spatial reasoning, not only in a general sense (e.g., its development) but also in terms of the particular manners in which children's spatial reasonings occur. In response to this call, we examine the role that drawing plays in the growth of children's conceptual understandings concerning properties and spatial relationships within and amongst two-dimensional shapes as well as the three-dimensional world.

Currently, prominent perspectives assume children's drawings to be external representations of their inner mental functioning. As such, establishing a child's level of understanding is achieved by comparing drawings against a hierarchical set of developmental and cognitive criteria (e.g., Piaget & Inhelder, 1956; Mulligan & Mitchelmore, 2009). Our research, in contrast, takes an alternative approach. We conceive children's drawings to be ever-emergent artefacts and multi-sensory motor processes of *thinking forth of a world or worlds* (Woodward, 2012). In this way, drawing is "a matter of learning as much as it [is] a matter of thinking" (Cain, 2010, p. 32). Thus, any deep insights we might gain about children's spatial reasonings necessitate inquiry into the moment to momentness of their acts of drawings and the conceptualizations that occur within these moments.







Our research reveals instances in which drawing plays diverse and compelling roles in children's spatial reasonings. Through examplars taken from case studies across kindergarten through the second grade, we illustrate: how the children come to draw as a mode of thinking; the different ways that they draw and use drawings to attend to important mathematical ideas (Depraz, Varela & Vermersch, 2003); and the conceptions that arise with and in drawing that contribute substantially to the growth of their geometric and spatial reasonings.

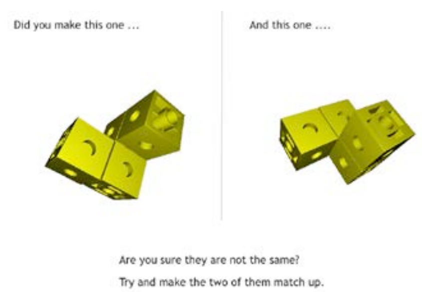
Use of the iPad as a mediator for the development of spatial reasoning

Catherine D. Bruce (Trent University)

Given the recent explosion of handheld technologies and tablets in education, researchers Bruce, Davis, Moss and Sinclair of the IOSTEM spatial reasoning working group have been investigating the role of iPad technology as a mediator of spatial reasoning for young children. There are three aspects to these explorations: i) Identifying the visual-spatial affordances of the iPad; ii) Capitalizing on the use of gesture to increase spatial reasoning; and, iii) 'Spatializing' mathematics, within but also beyond geometry to mathematics domains such as number sense and numeration. Design Research provides a useful research methodology and framework for iPad product development because it emphasizes development-testing-refinement-testing, and embraces complex contexts such as mathematics classroom and home learning environments.

Visual-spatial affordances of the iPad

In the four-cube challenge iBook generated by Bruce, children are encouraged to use interlocking cubes to make all possible combinations of four cubes. When checking to see if they have found the comprehensive set, they can look at pre-generated figures on the iPad screen and use a directional tracing gesture in the direction they wish to spin the figure, which makes the figure rotate on the screen.



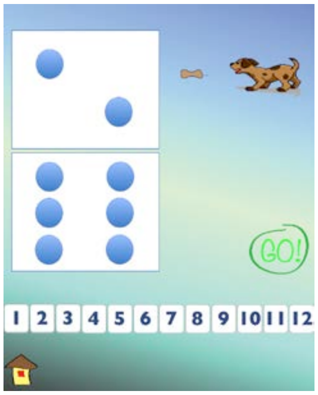
This enables rotation for visual comparing of interlocking cube figures to screen figures by aligning orientation of real and pictorial images, but it also enables comparison of multiple figures on the iPad for congruence. The power of the visual images, combined with their dynamic properties pushes children to consider the similarities and differences of mirror figures.

Capitalizing on the use of gesture when using the iPad

Compared to handling physical tools, engaging with virtual tools on an iPad provides students with *different* kinesthetic experiences. Given that much of the input on an iPad directly employs one's fingers, the potentially unique role of gestures within the interactive environment of tablet technologies is worth exploration. Sinclair and Jackiw's (2011) *TouchCounts* application, takes advantage of gesture by having users perform actions with their fingers that mirror the mathematics they are engaging in. For example a two finger pinching gesture brings circles of quantities together in order to add them together - If the child has 3 dots in one circle and 4 dots in a second circle, and then pinches these together (frame 3), the sets are joined to make one circle of 7 dots (frame 4).

iii) Spatializing Number through Apps

Spatial arrangements that are non-verbal and illustrate quantity in organized and familiar structures, help students build fluency with quantity and addition.



In the iPad game Spot the Dots, developed by Bruce, children may reason via spatial magnitude information, that the quantity in the lower square (see screen capture) is greater than the top square. The arrangement of dots in the bottom square consists of two rows of three (6). The child may count-on from 6, to arrive at 8. Alternately, the child may see two columns of 4 dots, treating the squares as one figure.

Robotics and Spatial Reasoning

Krista Francis (University of Calgary)

In anticipation of upcoming changes to include spatial reasoning in elementary mathematics standards and curriculum, identifying and describing spatial reasoning in educational contexts is crucial for making informed decisions. Most of what is known about spatial reasoning comes from psychology (see Casey, Dearing, Vasilyeva, Ganley, & Tine, 2011; Kayhan, 2005; Levine, Huttenlocher, Taylor, & Langrock, 1999), where tasks are diagnostic measures. Educational tasks differ from psychological tasks because they not only require opportunities for teachers to assess ability (diagnostic), they also permit opportunities for students to learn and improve (develop). A study was designed to identify and observe spatial reasoning in an educational context in order to develop observational protocols and to begin imagining spatial reasoning curriculum outcomes.

This poster will present findings from this study. The robotics task aligned perfectly with the mathematical processes described in the front matter of the Alberta Program of Studies (Alberta Education, 2007): communication, connections, problems solving, reasoning, technology, and visualization. However, no specific outcomes aligned in any of the general outcomes: number, pattern, space and shape, or statistics and probability. Thus prompting the question: How might spatial reasoning outcomes be developed?

Study participants included 21 children and 5 teachers during a 4-day long Lego Mindstorms™ robotics camp. Data collected included videos of children building and programming their robots. Video data permitted opportunities for detailed observations and allowed the researchers to repeated view the robot building at both slower and faster speeds. Analysis of the video was based initially on Bruce et al.'s (2013) list of skills associated with spatial reasoning. Descriptions of observed spatial reasoning skills were compiled. The observations and descriptions formed an imagined spatial reasoning curriculum.

In this poster presentation, we will present our imagined curricula and show a short two-minute video clip of one boy completing Steps 8 and 9 in the Lego instruction booklet for building a robot.

Figure 1: Video of boy building a robot at http://www.ucalgary.ca/IOSTEM/files/IOSTEM/video1-spatial_reasoning-480.mov

The video exemplifies the imagined spatial reasoning outcomes and the how the boy engaged in multiple spatial reasoning skills almost concurrently. Rather than isolated, sequential and fragmented, the boy repeatedly cycled through many spatial reasoning skills. The observation of the cyclical engagement of spatial reasoning skills highlights a caution for simply adding specific outcomes to the existing Program of Studies. At risk is a fragmentation of integrated process of spatial reasoning skills. The video also illustrates how the educational task permitted assessment of the boy's capabilities with rotation (a spatial skill) and his ability to learn. Poster observers will have opportunities to try to complete building the same Steps 8 and 9 of the robot.

A Misnomer No More: Using *Tangible* Cube-Figures to Measure 3D Mental Rotation in Young Children

Zachary Hawes (University of Toronto) & Catherine D. Bruce (Trent University)

In studies with adolescents and adults, three-dimensional (3D) mental rotation skills have proven to be a powerful predictor of mathematics achievement (Casey, Nuttall, Pezaris, & Benbow, 1995; Tolar, Lederberg, & Fletcher, 2009; see Figure 1). However, little is known about the development of 3D mental rotation in young children and even less is known about how the skill might relate to later mathematics learning (Mix &

Cheng, 2012). This lack of knowledge is likely a result of developmentally inappropriate testing paradigms (Hoyek, Collet, Fargier, & Guillot, 2012). The primary objective of this study was to design a measure of 3D mental rotation appropriate for children aged 4-8. Our task differs from traditional measures because it uses *tangible* block figures, is not time-limited, and provides a coloured 'anchor' cube to reduce executive function demands. A second objective was to examine the onset and development of 3D mental rotation in young children.

Methods

165 children (94 boys) between the ages of 4 and 8 participated (M = 6.0 years, SD = .9, range = 4.3 to 8.0 years). The 3D Mental Rotation Block Task (3D-MRBT) consisted of one practice item and 16 test items (Fig. 2). For each item, participants were presented with a target figure and three response figures, one of which was a perfect replica of the target figure but positioned in a different orientation. Participants were asked to indicate the figure that could be rotated to match the target (Fig. 3).

Results

Performance on the 3D-MRBT task was significantly correlated with age $r(161) = .44$, $p < .001$ (see Figure 4). One-sample *t*-tests were carried out for each age group to assess performance above chance, defined as 5.33 (i.e., 16 items divided by 3 answer choices). With the exception of the youngest age group, $t(6) = .30$, $p =$ *n.s.*, all other age groups performed above chance, $p < .05$. These data indicate the ability to mentally rotate 3D figures emerged between 4 ½ to 5 years of age and performance improved linearly as a function of age.

Discussion

To our knowledge, this is the first test of 3D mental rotation that (1) uses tangible figures to assess the skill, and (2) demonstrates that children as young as 4 and ½ are capable of 3D mental rotation. The early onset of 3D mental rotation reported here contrasts the late onset reported by other researchers using traditional measures of 3D mental rotation (e.g., see Hoyek et al., 2012).

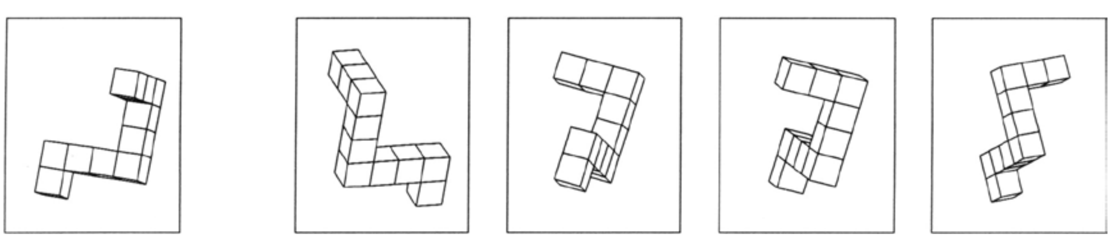


Figure 1. An example item from the Vandenberg and Kuse (1978) 3D mental rotation task. Participants are presented with a target item (far left) and four response items.

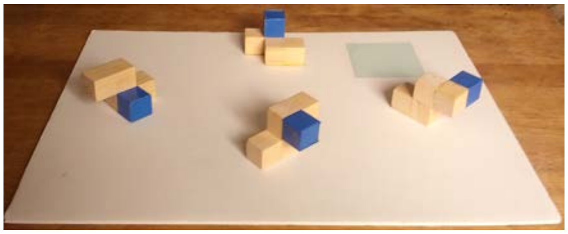


Figure 2. Example of an item from the 3D mental rotation block task (3D-MRBT)

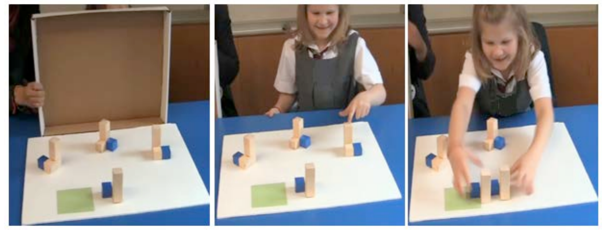


Figure 3. 3D-MRBT: To begin, each test item is shielded from view. Participants are then asked to look carefully at the three options and point to the item that matches the target. Once an item has been selected, participants are asked to place the item next to the target and show how it can be rotated to match.

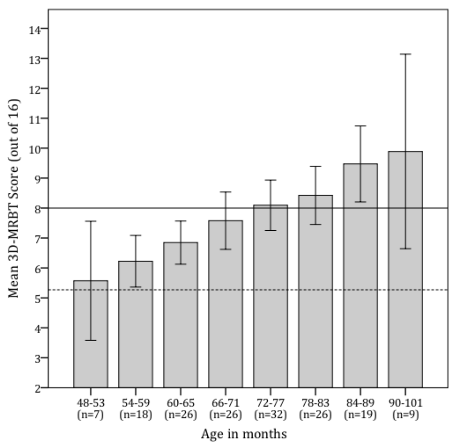


Figure 4. Performance by age; the dotted line denotes chance (33% correct) and the solid black line denotes a conservative estimate of 3D mental rotation ability based on performance at or above 50% correct. Bars represent 95% confidence intervals around each mean.

"They are getting married!" Towards a dynamic, functional understanding of symmetry in primary school

Oi-Lam Ng, Simon Fraser University

Our goal in the current study is to discuss the potential for children to develop new forms of thinking about symmetry with teaching interventions, drawing on the dynamic nature of DGEs. In response to Bryant's call for more intervention studies to be done in the area, we are interested in examining how children's understanding of symmetry evolves within a teaching experiment. In addition, our work with DGEs has prompted us to try to better understand the effects that such a digital technology might have on children's thinking of symmetry, particularly in relation to its dynamic nature. What does teaching children with a dynamic approach of symmetry look like? What effect does a DGE have on children's thinking about symmetry?

We use Sfard's communicational framework to study children's discourse (thinking) while they engage in a sequence of lessons on symmetry. In addition, we focus on children's word use, gestures, and use of diagrams during the lessons involving interaction of dynamic visual mediators. Within this theoretical perspective, our aim is to study how the dynamic environment changes the way the children think of symmetry and to identify the particular tools that serve as instruments for semiotic mediation in their learning.

This teaching experiment involves three lessons, each occurring two weeks after the previous one, taught in an elementary school in Western Canada. Each lesson was taught to two different groups of children (a grade 1/2 split and a grade 2/3 split—each having about 22 students) and lasted approximately one hour. Each lesson included both computer-based activities as well as pencil-and-paper activities. Lessons 1 and 2 involved interactions with the discrete symmetry machines sketches shown in Figure 1.

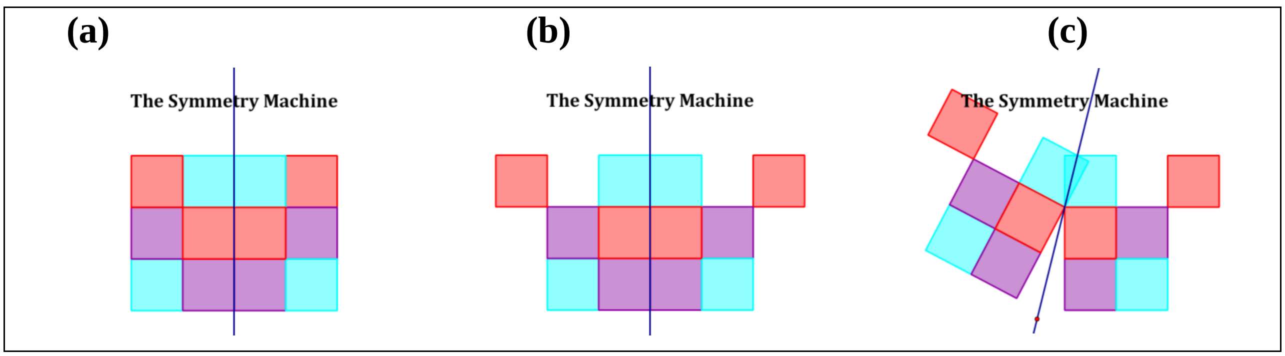
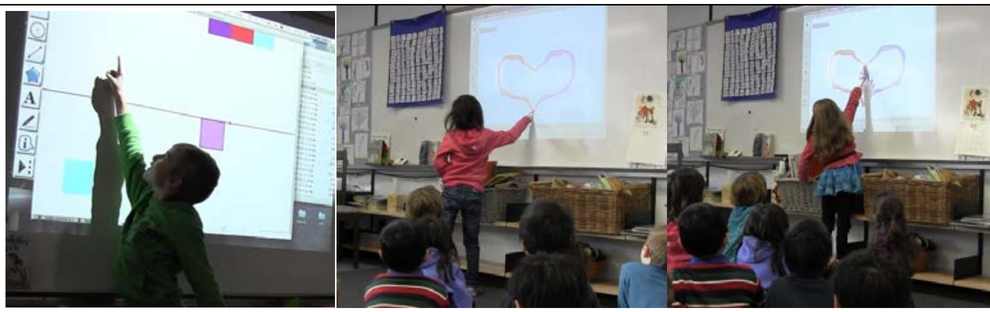


Figure 1: (a) The discrete symmetry machine; (b) After dragging one block away from the line; (c) After rotating the line of symmetry

Lesson 3 involved interaction with the continuous symmetry machine sketch shown in Figure 2. This is a blackbox sketch in which dragging a point causes the other point to move—in this case, in such a way that it remains symmetric with respect to a hidden vertical line of symmetry.

Over the course of the three lessons, which included a large component of whole class discussion and interaction with the projected images in *Sketchpad*, as well as opportunities for the children to create drawings based on both the discrete and continuous symmetry machine, the children changed their thinking about symmetry. They began with a static discourse on symmetry that was focused on the intrafigural qualities of shapes and that featured a small example space of shapes with vertical reflectional symmetry. The children began to talk about interfigural qualities, focusing on the functional relationships of a pre-image and its image. This shift was occasioned by the processes of semiotic mediation in which the dragging tool, as well as the language and gestures of the teacher, became signs that enabled communication about central features of reflectional symmetry including: the way in which one side of a symmetric design is the same as the other; the way in which one component of a symmetric design is the same distance away from the line of symmetry as its corresponding image; the way in which a pre-imagine component and its image have to be on the same line relative to the line of symmetry; and, the way in which a pre-image and an image gives rise to parity.



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